On A Non-Archimedean Broyden Method

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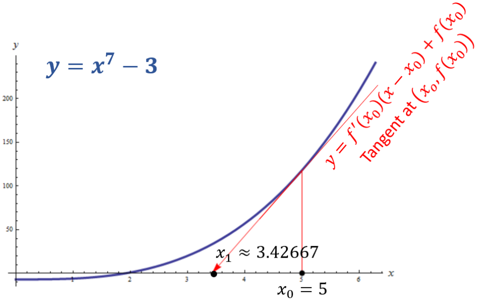
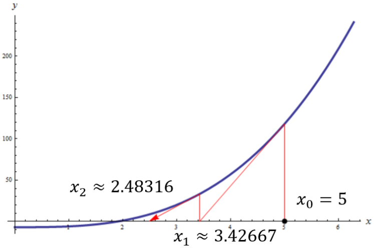
Outline

Newton method over $\mathbb{R}$





Approximation of a simple root $x^\star$ $(f'(x^\star) \neq 0)$.

Start from a guess x_0, and compute $\boxed{x_{n+1} = x_n - f'(x_n)^{-1} f(x_n)}$.

System of m equations F in m unknowns: $x^\star \in \mathbb{R}^m$, $F(x^\star)$, $\mathrm{Jac}_F(x^\star)$ invertible

Start from a guess x_0 and compute: $\boxed{x_{n+1} = x_n - \mathrm{Jac}_F(x_n)^{-1} F(x_n)}$ in $\mathbb{R}^m$.

Main result

- In 1965, Broyden suggested to replace the Jacobian matrix $\mathrm{Jac}_F(x_n)$ by an approximation B_n of it $\rightarrow$ quasi-Newton method

$$x_{n+1} = x_n - B_n^{-1} F(x_n), \quad x_n, x_{n+1} \in \mathbb{R}^m, \quad B_n \in \mathrm{Mat}_m(\mathbb{R}^m), \quad B_n \approx \mathrm{Jac}_F(x_n)$$

- He chose $B_0 \approx \mathrm{Jac}_F(x_0)$, then for B_1 a 1-dimensional deformation of B_0, then for B_2 a 1-dim deformation of B_1 etc.
- 1965-today: numerous improvements, variants and generalizations...
- ...but nothing for systems with coefficients in a complete valued field (p-adic or power series coefficients).

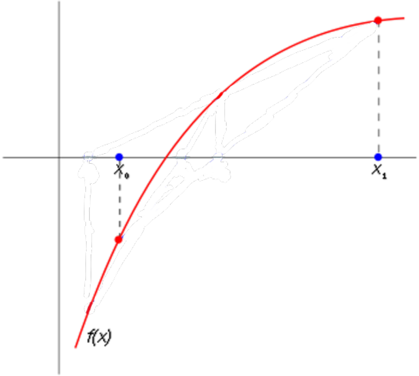
Main outcome : adapt the Broyden method to a system F with coefficients in a valued complete field $\rightarrow$ first non-archimedean quasi-Newton method

Broyden method in dim 1: secant method

Broyden method generalizes the secant method (dimension 1) to dimension m.

Approximation of simple root $x^\star$
$f(x^\star) = 0, \quad f'(x^\star) \neq 0$
Start with two guesses x_0, x_1 then,

$$x_{n+1} = x_n - \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} f(x_n).$$



Let $B_n^{-1} = \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} \quad \rightarrow \quad \boxed{B_n(x_n - x_{n-1}) = f(x_n) - f(x_{n-1})}$

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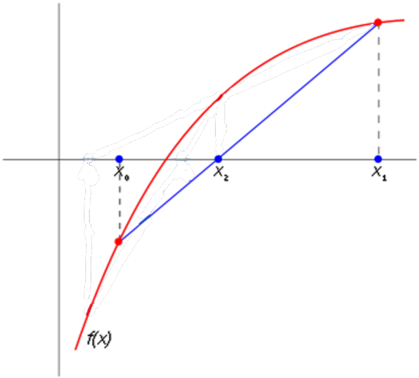

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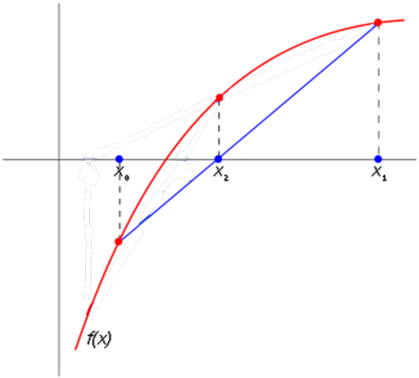

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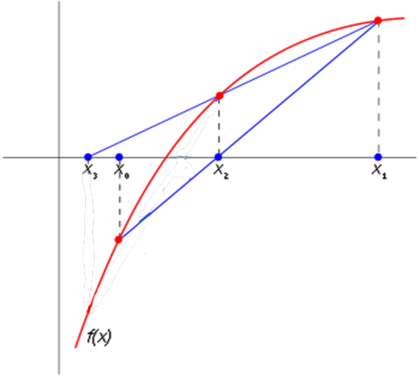

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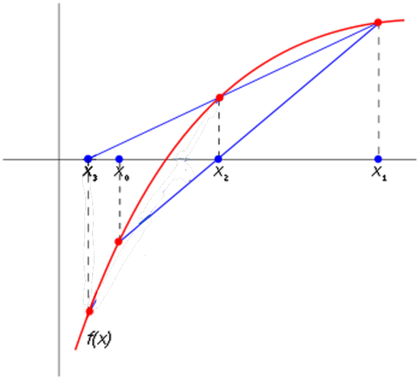

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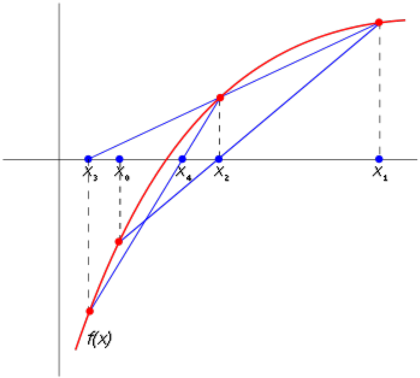

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Broyden's construction ($F : \Omega \subset \mathbb{R}^m \to \mathbb{R}^m$, $F(x^\star) = 0$, $\mathrm{Jac}_F(x^\star) \neq 0$)

- Start with one guess x_0 and compute $B_0 \approx \mathrm{Jac}_F(x_0)$.
- Then $x_{n+1} = x_n - B_n^{-1} F(x_n)$ with B_n verifying:
  - $B_n(x_n - x_{n-1}) = F(x_n) - F(x_{n-1})$ (Rewrite it $B_n s_{n-1} = y_{n-1}$)
  - $B_n z = B_{n-1} z$ for all z orthogonal to s_{n-1}.
- These two conditions suffice to characterize B_n from B_{n-1}, namely:

$$B_n = B_{n-1} + (y_{n-1} - B_{n-1} s_{n-1}) {u_{n-1}}^T, \quad \text{where} \quad {u_{n-1}}^T s_{n-1} = 1$$

- Broyden's choice : $u_{n-1} = s_{n-1} / \|s_{n-1}\|^2 = s_{n-1} / {s_{n-1}}^T \cdot s_{n-1}$.

Theorem 1 (Sherman-Morrison)

Computation of the inverse B_n^{-1} from B_{n-1}^{-1} can be done in $\approx 5m^2$

$$B_n^{-1} = B_{n-1}^{-1} - \frac{B_{n-1}^{-1} F(x_n) {u_{n-1}}^T B_{n-1}^{-1}}{{u_{n-1}}^T B_{n-1}^{-1} y_{n-1}}$$

Presentation of the Broyden method over $\mathbb{R}$

Start with $B_0 \approx \mathrm{Jac}_F(x_0)$ for x_0 near to a non-singular solution $x^\star$ of F.
Broyden update (Iteration n)

$$\begin{aligned} x_{n+1} &= x_n - B_n^{-1} F(x_n), \\ B_{n+1}^{-1} &= B_n^{-1} - \frac{B_n^{-1} F(x_{n+1}) {u_n}^T B_n^{-1}}{{u_n}^T B_n^{-1} y_n}, \end{aligned} \quad \text{(vector } u_n \text{ verifying } {u_n}^T s_n = 1, \text{ like } u_n = \frac{s_n}{\|s_n\|^2} \text{)}$$

Cost of the n-th iteration (machine precision):

- $\approx 6m^2$ operations, and m evaluations of a scalar function (one evaluation of F)

Theorem 2 (Broyden — Dennis — Moré, 1973)

If F is sufficiently regular and x_0 sufficiently close to $x^\star$ then Broyden method produces a sequence that converges superlinearly to $x^\star$: $\boxed{\lim_{n\to\infty} \frac{\|x_{n+1} - x^\star\|}{\|x_n - x^\star\|} = 0}$.

Advantage of quasi-Newton methods over $\mathbb{R}$

Newton method:

- Computing the Jacobian matrix not always obvious... (1965)
- Needs to evaluate the m^2 entries of the Jacobian at each iteration.
- Needs to solve the system $\mathrm{Jac}_F(x_n)s_n = -F(x_n)$ (in s_n) requires $O(m^3)$ (or $O(m^\omega)$ operations for m large).
- Quadratic convergence is fast, yet the region of quadratic convergence may be very small.

Whereras, with a Broyden update:

- Update costs $\approx 6m^2$ and one evaluation of F.
- $\rightarrow$ no need to evaluate m^2 entries at each iteration.
- But has a slower rate convergence, that deteriorates with m.

However this drawback is mitigated with machine precision.

Outline

Adaptation to the non-archimedean setting (ex: p-adic or $k[[x]]$)

- With Broyden's original proposal: $B_{n+1} = B_n + \frac{(y_n - B_n s_n) {s_n}^T}{{s_n}^T \cdot s_n}$
  $(s_n = x_{n+1} - x_n, \qquad y_n = F(x_{n+1}) - F(x_n))$
- requires a dot product which is an inner product over $\mathbb{R}^m$
  - Warning: dot product is isotropic over $\mathbb{Q}_p$.
  - However, if we use the previous formula:
    $B_{n+1}^{-1} = B_n^{-1} - \frac{B_n^{-1} F(x_{n+1}) {u_n}^T B_n^{-1}}{{u_n}^T B_n^{-1} y_n}, \quad {u_n}^T s_n = 1$ we only need ${u_n}^T B_n^{-1} y_n \neq 0$.

Contribution 1 (Adaptation)

Broyden update is adaptable over any complete valued field.

Idea: Write $s_n = (\sigma_1, \ldots, \sigma_\ell, \ldots, \sigma_m)^T$ and assume: $\mathrm{val}(\sigma_\ell) = \min_{i=1,\ldots,m} \mathrm{val}(\sigma_i)$, ℓ is the smallest index verifying this property. Let $u_n = (0, \ldots, \sigma_\ell^{-1}, \ldots, 0)^T$. Then naturally $1 = {u_n}^T s_n$, and for technical reasons ${u_n}^T B_n^{-1} y_n \neq 0$.

Intermezzo: ultrametric norms

Let K be a complete discrete valued field:

- $K = \mathbb{Q}_p$, p-adic field or $K = k((X))$ field or Laurent series over a field k.
- Absolute value:
  - $|a| = p^{-\mathrm{val}(a)}$ when $a \in \mathbb{Q}_p$, and $|a| = 2^{-\mathrm{val}(a)}$ when $a \in k((X))$.
  - ultrametric inequality: $|x + y| \leq \max\{|x|; |y|\}$ (equality if $|x| \neq |y|$).
- Standard p-adic norms: $\|\overrightarrow{x}\| = \max\{|x_1|_p, \ldots, |x_m|_p\}$.
  - Approximation x_n of a solution $x^\star$: $\|F(x_n)\|$ is closer and closer to zero.

Lemma 3 (Operator norm on a matrix A)

$\|A\| = \max_{\|x\|=1} \|Ax\|$ *is actually equal to the max-norm:* $\|A\| = \max\{|\mathrm{entries\ of}\ A|\}$.

- Over $\mathbb{R}$, matrices B_n minimize the Frobenius norm $\|B_{n+1} - B_n\|_F$
- $\rightarrow$ Has been used to simplify the proof of superlinear convergence $\rightarrow$ no such norm in the non-archimedean setting.

R-superlinear convergence

Contribution 2 (Convergence)

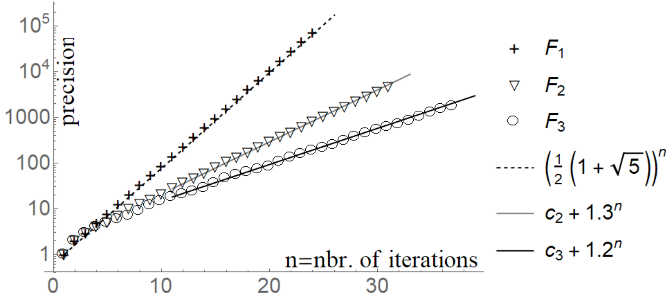
the non-archimedean Broyden method converges R-superlinearly of order at least $\mu = 2^{1/2m}$:

$$\limsup_{k\to\infty} \|x_k - x^\star\|^{1/\mu^k} < 1$$

- Meaning: The sequence $\|x_k - x^\star\|$ is not necessarily strictly decreasing. Essentially one can "think" that after $2m$ steps
  - $\|x_{n+2m} - x^\star\| \leq C\|x_n - x^\star\|^2$, for a constant C.
- Experimental observations suggest rather an "almost" Q-superlinear convergence of order $\alpha \approx 2^{1/m}$ $\rightarrow$ $\lim_{n\to\infty} \frac{\|x_{n+1}-x^\star\|}{\|x_n-x^\star\|^\alpha} \leq r$
- Hypotheses: $\mathrm{Jac}_F(x^\star)$ is invertible, and F verifies a kind of strong form of Taylor expansion at order 2 in a neighborhood U of $x^\star$:
  - $\|F(y) - F(x) - \mathrm{Jac}_F(x^\star)(y-x)\| \leq c_0\|y-x\|^2, \quad \forall x, y \in U, \quad c_0 > 0$
- Remark: This is the longest and most technical part of the article.

Experiments

F_1: 2 polynomials, 2 unknowns
F_2: 3 polynomials, 3 unknowns
F_3: 4 polynomials, 4 unknowns



For $n = 2$, the order of superlinear convergence is $\Phi = \frac{1}{2}(1+\sqrt{5})$: like in the secant method in one variable (archimedean setting or not).

- [E. Bach] Iterative root approximation in p-adic numerical analysis. *J. of Complexity* 2009

Implementation at finite precision: comparison with Newton

Here, the arithmetic cost depends on the precision. In case of Newton method (quadratic convergence):

$$\underbrace{x_{n+1}}_{2^{n+1}\ \text{“digits”}} = \underbrace{x_n}_{2^n\ \text{“digits”}} - \underbrace{\mathrm{Jac}_F(x_n)^{-1}F(x_n)}_{\text{“digits” in the interval } [2^n,2^{n+1}]} .$$

- Therefore, no digits overlap in the sum $\to$ easy to implement and analyze.
- $\to$ The ratio (speed of convergence)/(precision gained) is somewhat optimal.
- Update of the Jacobian's inverse (by Newton method applied to $A \mapsto A^{-1}$)

$$\mathrm{Jac}_F(x_{n+1})^{-1} = 2\mathrm{Jac}_F(x_n)^{-1} - \mathrm{Jac}_F(x_n)^{-1}\mathrm{Jac}_F(x_{n+1})\mathrm{Jac}_F(x_n)^{-1}$$

- Cost: $O(m^\omega)$ for matrix products.
  $O(mL)$ for the evaluation $\mathrm{Jac}_F(x_{n+1})$ ([Baur-Strassen, 1980] $\to$ if F is polynomial and can be valuated in L operations).

Implementation at finite precision: management of precision

Contribution 3 (Analysis of an implementation at finite precision)

Implementation, with polynomials as input, of Broyden method over $\mathbb{Q}_p$ and $k((X))$ at finite precision and complexity analysis.

- Difficulty 1: The Broyden update uses a division $\rightarrow$ bad for the precision

$$B_{n+1}^{-1} = B_n^{-1} - \frac{B_n^{-1} F(x_{n+1}) u_n^T B_n^{-1}}{u_n^T B_n^{-1} y_n}, \quad x_{n+1} = x_n - B_n^{-1} F(x_n)$$

  $\rightarrow$ can be addressed by tracking p-adic intervals along these operations.
- Difficulty 2: convergence is not quadratic, and is not known.
  - Issue: If we extend precision of interval arithmetic too much, we loose efficiency.
  - however, the speed of convergence ressembles to superlinear with an order of convergence α after a few iterations $\rightarrow$ allows to guess without too much loss.

Complexity analysis - Comparison with Newton

- Notations:
  - $\mathsf{M}(d)$ cost of multiplying two truncated power series/p-adic integers in an interval of length d.
  - L number of arithmetic operations to evaluate the system F at any vector.
- n-th iteration of Newton method (interval of length 2^n): $O(\mathsf{M}(2^n)(m^\omega + mL))$
- $\rightarrow$ to reach precision $N \approx 2^\ell$, $O(\mathsf{M}(N)(m^\omega + mL))$.

Assumption (order of superlinear convergence α for Broyden method)

Assume $\|x_{n+1} - x^\star\| \leq r\|x_n - x^\star\|^\alpha$ for $\alpha > 1$ and a constant $r > 0$.

- Cost of one iteration: $O(\mathsf{M}(\alpha^n)(L + m^2)) \Rightarrow$ $\boxed{O(\mathsf{M}(\frac{N}{\alpha-1})(m^2 + L))}$.
- If we assume $\alpha \approx 2^{1/m}$ then $O(\mathsf{M}(Nm)(m^2 + L))$ which is worse than Newton.

Application - Future work

Remark: Relaxed arithmetic [van der Hoeven et al.] (specific to Newton operator).

- p-adic and power series coefficients: quadratic convergence while maintaining $O(\mathrm{M}(N)(m^2 + mL))$

Not better than Newton. . . so what's the point ?

1. Derivative-free: Jacobian is not easily accessible/computable $\rightarrow$ divided-difference matrix.
2. Non-polynomial functions: [Baur-Strassen, 1980]'s theorem does not hold $\rightarrow$ Jacobian is complicated to evaluate, may require up to m^2L operations to evaluate (instead of $O(mL)$).
3. Infinite dimensional problem: no Jacobian. Broyden's framework allows to work with finite dim. approximation [Kelley-Northrup,1988] [Kelley-Sachs, 1990] etc.