Fast construction of a lexicographic Gröbner basis of the vanishing ideal of a set of points

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Introduction

- Setting: Let $V \subset \overline{\mathbf{k}}^n$ a finite set of points
  V is Zariski-closed over $\mathbf{k}$: V is the set of solutions of a polynomial system over $\mathbf{k}$.
- Problem: Compute a lexicographic Gröbner basis of the vanishing polynomials on V.
  - Classical problem: Buchberger-Möller (1982) for any monomial order
  - …and for the lex order, dedicated algorithms: 1995 to 2016.

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  - ...and for the lex order, dedicated algorithms: 1995 to 2016.
- Yet, all those works are somewhat still "incomplete". Why ?
  - research articles tend to address some aspects and ignoring some others.
  - for example, fully explicit interpolation formulas have not appear clearly...
  - ...it is a key for a sharp complexity study.

Non-generic LexGB of Dimension Zero

LexGB = Lexicographic Gröbner Basis: $x_1 \prec x_2 \prec \cdots \prec x_n$.

$$
\begin{aligned}
g_{\ell(n)}(x_1, x_2, x_3, \ldots, x_{n-2}, x_{n-1}, x_n) &= x_n^{d_{\ell(n)}} + \cdots \\
g_{\ell(n)-1}(x_1, x_2, \ldots, x_{n-2}, x_{n-1}) &= \mathrm{lc}_{n-1}(g_{\ell(n)-1}) x_n^{d_{\ell(n)-1}} + \ldots \\
\ddots \quad &\vdots \quad \vdots \\
g_{\ell(n-1)}(x_1, \ldots, x_{n-1}) &= x_{n-1}^{d_{\ell(n-1)}} + \cdots \\
\ddots \quad &\vdots \quad \vdots \\
g_{\ell(2)}(x_1, x_2) &= x_2^{d_{\ell(2)}} + \cdots \\
g_{\ell(2)-1}(x_1, x_2) &= x_1^{n_{\ell(2)-1}} x_2^{d_{\ell(2)-1}} + \cdots \\
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g_1(x_1) &= x_1^{d_1} + \cdots
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This work $\rightarrow$ "Highly" non-generic lexGB : $|\mathcal{G}| > n$.

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Non-generic: Shape Lemma, Triangular Set $\rightarrow$ nothing new.

Results

Let $D_i := |V_{\leq i}|$ where $V_{\leq i} = \pi_i(V)$,
$\pi_i : \overline{\mathbf{k}}^n \to \overline{\mathbf{k}}^i, (a_1, \ldots, a_n) \mapsto (a_1, \ldots, a_i)$

1. There is a Gröbner basis $\mathcal{G}'$, non-reduced in general, such that any polynomial $g \in \mathcal{G}'$ can be computed in at most:

$$O(\mathrm{A}(D_1) + \mathrm{A}(D_2) + \cdots + \mathrm{A}(D_n)) < O(nD_n),$$

arithmetic operations in $\mathbf{k}$.

- $\mathrm{A}(d)$ cost to construct Lagrange idempotents of d points.
- $\mathrm{A}(d) = \mathrm{M}(d)\log(d)$, by the subproduct tree technique. ($\mathrm{M}(d) = O(d\log(d)\log\log(d))$ by Schönhage-Strassen, or naively d^2).

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2. The structure of (non-generic) lexGB allows to recycle computations.
   - But difficult to estimate in general. Simple strategy is still a work in progress.

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The lextree: introduction and backgrounds

A key tool to study lexGB is a combinatorial decomposition of V:
One-one correspondence between standard monomials of $\mathcal{G}$ and points of V.

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One-one correspondence between standard monomials and leaves

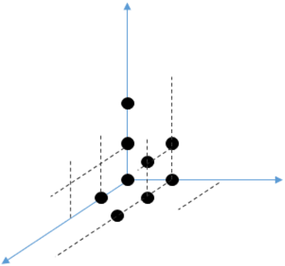
Lextree II

More than standard monomials, we are interested in leading monomials of a lexGB.

We introduce a new variation of computing standard monomials to compute leading exponents:

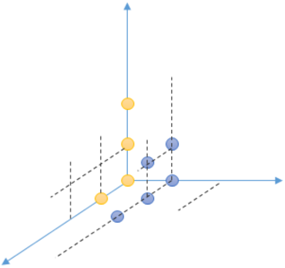
Example interlude

Lextree: construction



● root

Lextree: construction



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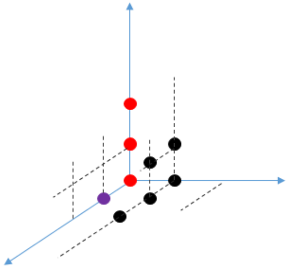

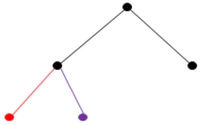

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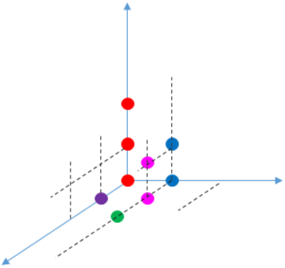

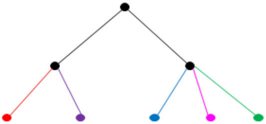

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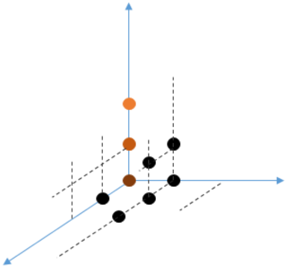

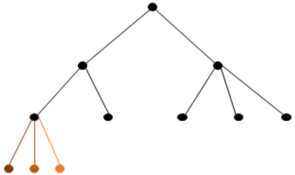

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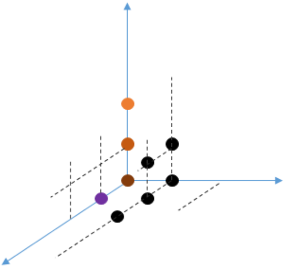

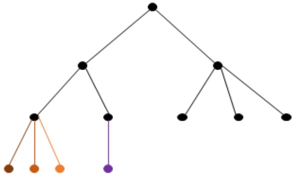

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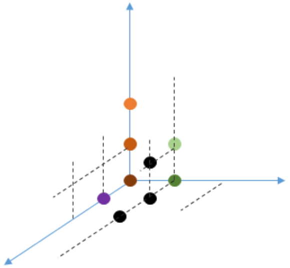

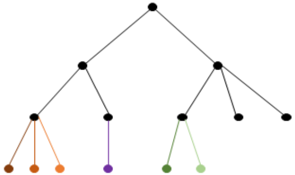

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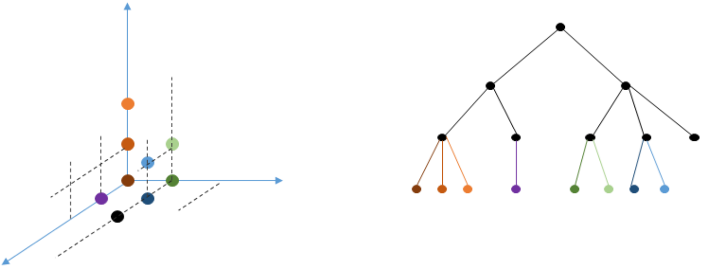
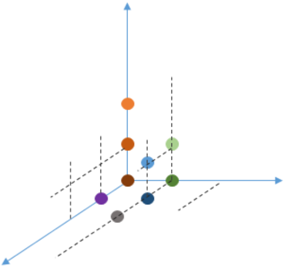
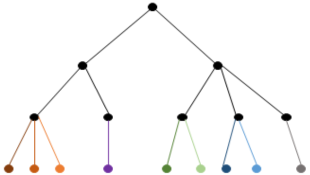


Lextree: From leaves to exponents in the GB





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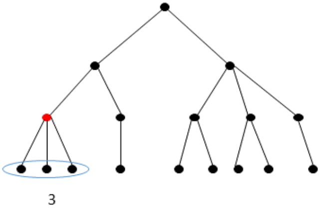

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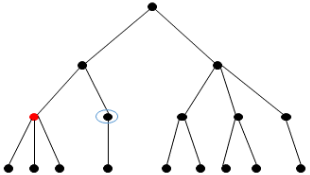

3 How many siblings with
> 3 children ?

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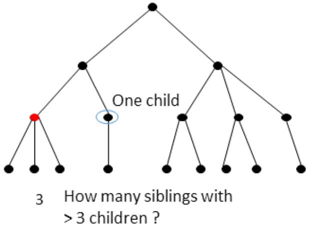

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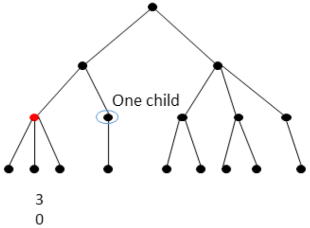

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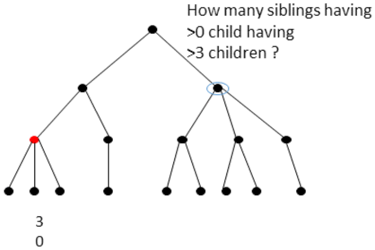

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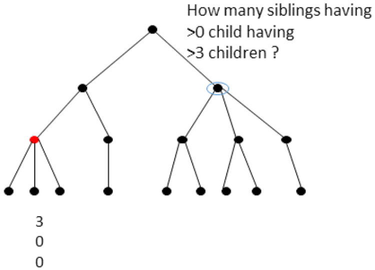

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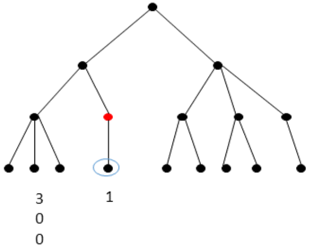

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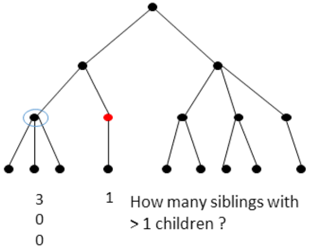

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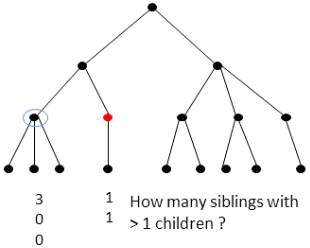

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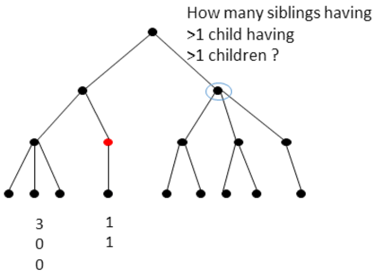

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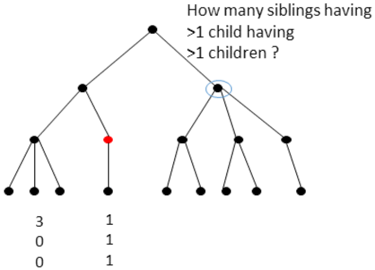

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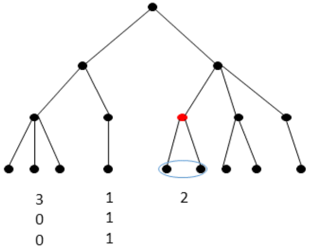

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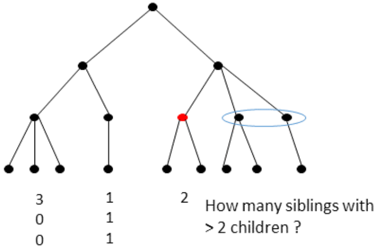

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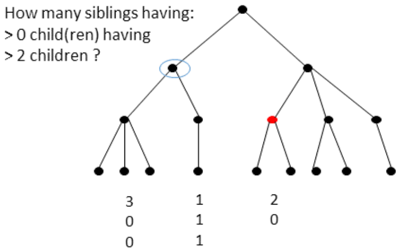

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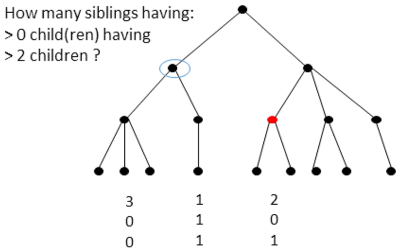

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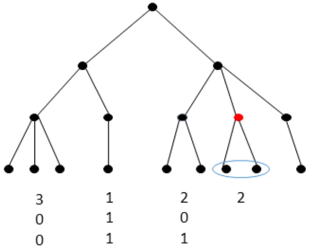

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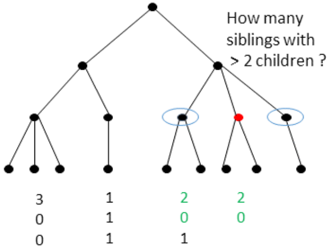

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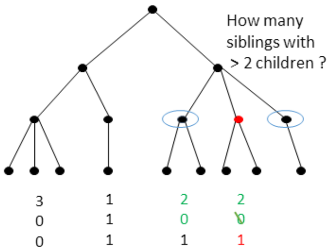

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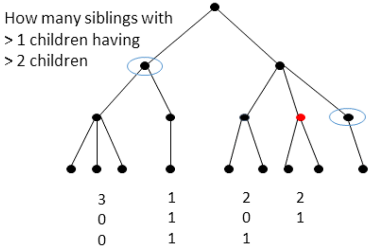

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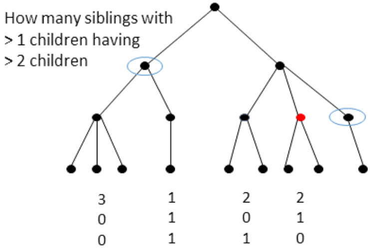

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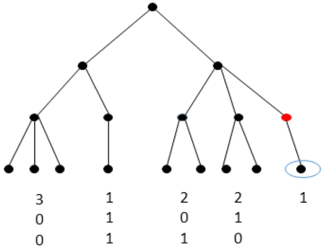

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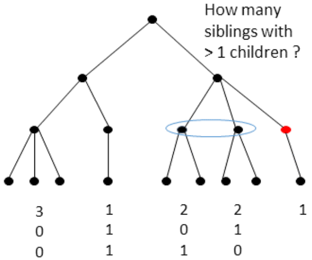

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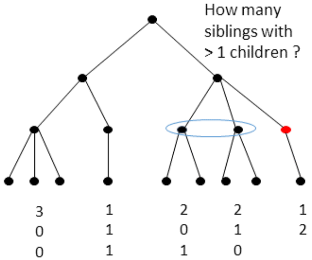

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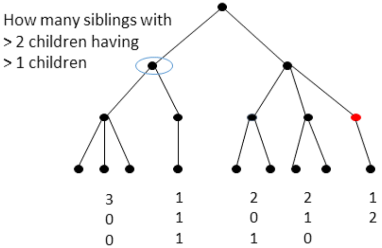

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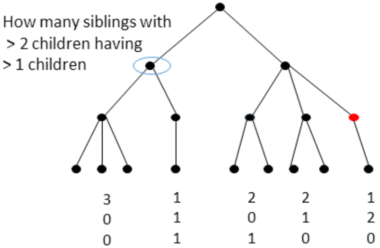

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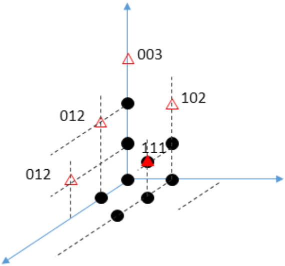

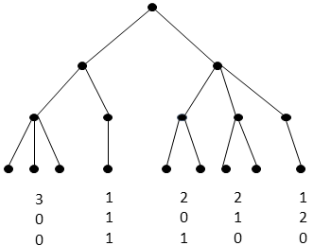

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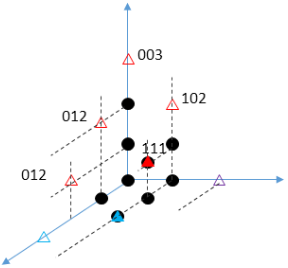

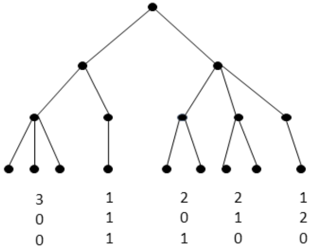

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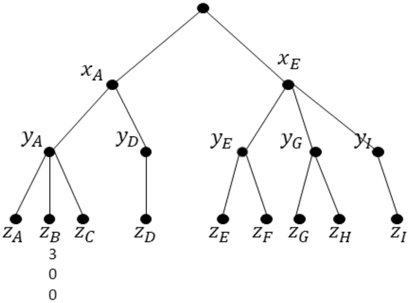

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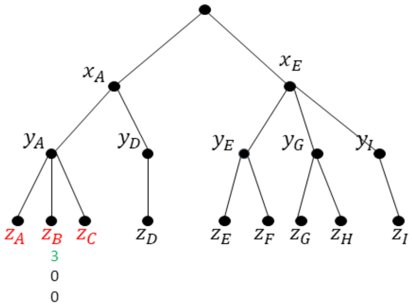

$(z - z_A)(z - z_B)(z - z_C)$ z^3

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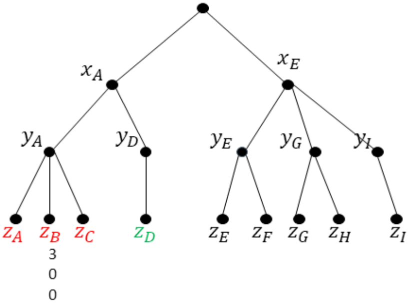

$$(z-z_A)(z-z_B)(z-z_C)\,\frac{y-y_D}{y_A-y_D} + (z-z_D)\,\frac{y-y_A}{y_D-y_A}$$

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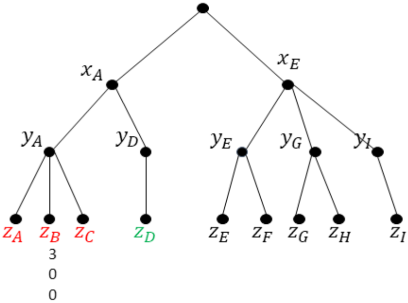

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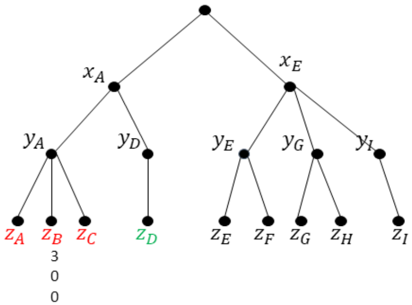

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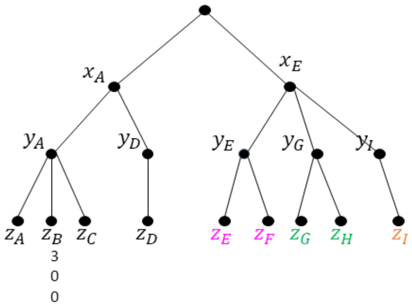

$(z - z_E)(z - z_F)\, z \quad + z(z - z_G)(z - z_H) \quad +$

$z^2(z - z_I)$

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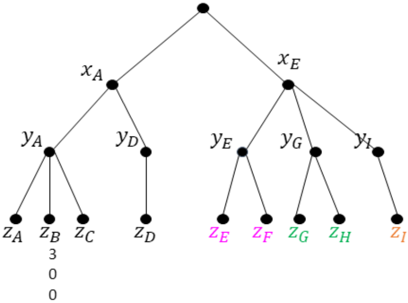

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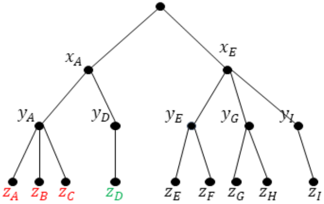

3
0
0

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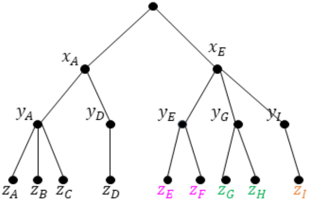

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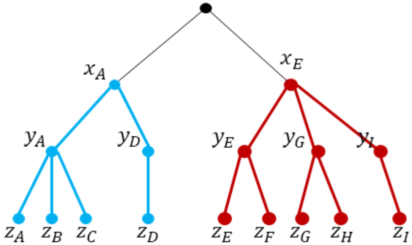

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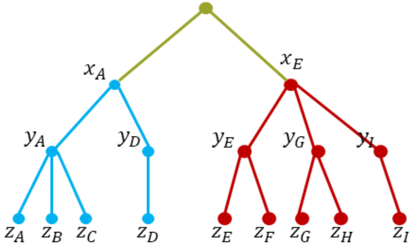

$$\left[(z-z_E)(z-z_F)\,z\,\frac{y-y_G}{y_E-y_G}\frac{y-y_I}{y_E-y_I} + z(z-z_G)(z-z_H)\frac{y-y_G}{y_E-y_G}\frac{y-y_I}{y_E-y_I} + z^2(z-z_I)\frac{y-y_G}{y_E-y_G}\frac{y-y_I}{y_E-y_I}\right] \times \frac{x-x_A}{x_E-x_A}$$

$$+ \left[(z-z_A)(z-z_B)(z-z_C)\frac{y-y_D}{y_A-y_D} + z^2(z-z_D)\frac{y-y_A}{y_D-y_A}\right] \times \frac{x-x_E}{x_A-x_E}$$

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Easy Fact: algebraic complexity depends only on the shape of the tree (number of children of nodes) and not on labels at each node.

Interpolating & Discarding points in the lextree

Given an exponent at a leaf $\mathbf{x^e}$, from (parent of the) leaf to the root, perform bottom-up test on siblings of the current node to identify:

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Non-reduced Gröbner basis $\mathcal{G}'$? Bit-size

- . . . but minimal: $\textsc{lm}(\mathcal{G}) = \{\textsc{lm}(g) \mid g \in \mathcal{G}\} = \textsc{lm}(\mathcal{G}')$.
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- More precisely let $h(g)$ be “roughly” the max bit-size among all coefficients on $g \in GB'$:

$$h(g) \leq O(nD^2 h_{pts}^2)$$

  where h_{pts} is the max bit-size among coordinates of all points in V.
- all in all, this is not a bad choice. . . it has moreover good properties. . .

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- Let $\phi_{\mathbf{a}} : \overline{\mathbf{k}}[x_1, \ldots, x_n] \to \overline{\mathbf{k}}[x_{m+1}, \ldots, x_n]$, $m < n$ evaluation map at $\mathbf{a} = (a_1, \ldots, a_m)$.
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  - whereas stability is: $\mathrm{LT}(\phi_{\mathbf{a}}(g)) \prec_m \phi_{\mathbf{a}}(\mathrm{LT}_m(g)) \Rightarrow \phi_{\mathbf{a}}(g) \equiv 0 \bmod \phi_{\mathbf{a}}(G \setminus \{g\})$.

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Previous work: Stability for *m*-elimination order (includes lex order)

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Summary of the interpolation of one polynomial

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Step 2. Bottom-up interpolation or discard sibling branches.

- This creates an arithmetic circuit. It depends only on the shape of the tree.
- In both case, subproduct tree can be used
$\rightarrow$ many times similar products must be perform.
- About the upper bound an arithmetic complexity $\rightarrow$
Can we do better ? Reuse already computed polynomials.

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The more polynomials in $\mathcal{G}'$, have been computed the more it is likely possible to recycle

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    - Work in progress: structural results. . .

Work in progress – Structure

Assume that $f \in \mathcal{G}'$, with $\mathrm{LM}(g) = x_1^{d_1} \cdots x_n^{d_n}$.

$$f = \sum_{\alpha \in A} \mathcal{L}_\alpha(x_1, \ldots, x_{n-1}) f_{1,\alpha}(x_1) \cdots f_{n,\alpha}(x_n) \cdot m_\alpha$$

- where $f_{i,\alpha}$ depends only on the $i-1$-th first coordinates $(\alpha_1, \ldots, \alpha_{i-1})$ of α,
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Kind of generalization of Lazard's structural theorem (1985) full result for lexGB in two variables.

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   - Lei-Zheng-Ruen 2014: investigated using Lederer formulation (four-in-a-row). From their own account, dautingly complicated to estimate complexity in this way.
   - the key is simplicity. It is the case when the Hermite conditions are triangular: the highest order in the derivative of Taylor expansions appear to the largest (single) variable.