Cayley graphs based on octonions, and their implementation in MAGMA

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Introduction

Lubotzky-Philips-Sarnak, 1986-88 "Ramanujan graphs"
Combinatorica **8**:261-277, 1988

G. Margulis "Explicit group-theoretic constructions of combinatorial schemes and their applications for the construction of expanders and concentrators

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Large girth No small cycle (actual record)

- a classical problem in extremal graph theoery,
- with several applications: LDPC error-correcting codes
- metric embeddings etc

LPS Ramanujan graphs and quaternions

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- Construction possible (and not trivial)
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→ Implementation in Magma, check on “small” parameters the second eigenvalue and the girth of these graphs.

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So any interesting porperties?

- Conjecture: they are non-vertex transitive
- Difficulty: How to describe a non-trivial automorphism?

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  - $|$connnected components of $G| =$ multiplicity of λ_0

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- Let A be a commutative ring with units:

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- Let q be a prime $q \neq 2$,

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- There is a "nice" family $\mathscr{P}(p) \subset \mathbb{H}(\mathbb{Z})$ of $p+1$-quaternions of norm p such that:

$\mathscr{C}ay(\langle\mathscr{P}(\mathrm{p})\rangle\ ,\ \mathscr{P}(\mathrm{p}))$ is the $p+1$-regular tree.

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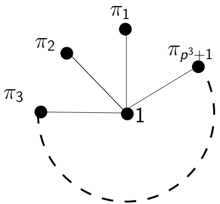

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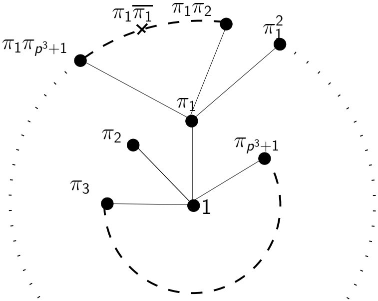

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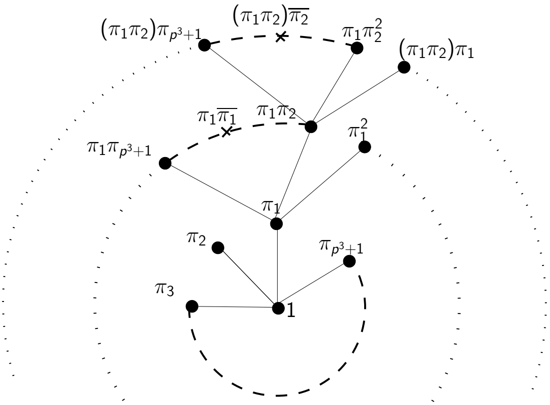

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- Let $\mathscr{S}(p,q) \equiv \mathscr{P}(p)$ mod q $(\mathscr{S}(p,q) \hookrightarrow \mathbb{H}(\mathbb{F}_q)^\star / \mathcal{Z} \simeq PGL_2(\mathbb{F}_q))$.

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To prove the remarkable properties: vertex-transitivity is essential

Outline of the new construction

Step 1 Infinite $p^3 + 1$-regular tree: used unique factorization of integral octonions in $\mathbb{O}(\mathbb{Z})$.

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For each prime $p > 2$, we get an infinite family $\mathscr{X}_p = \{\mathscr{X}_{p,q}\}_{q>p}$ of degree $p^3 + 1$-regular graphs.

Generalities on octonions

Let $\mathbb{O}(R)$ a free R-module of rank 8 with basis:

$$1\,,\ \mathrm{i}\,,\ \mathrm{j}\,,\ \mathrm{k}\,,\ \mathrm{t}\,,\ \mathrm{it}\,,\ \mathrm{jt}\,,\ \mathrm{kt},$$

such that $\mathbb{O}(R) = \mathbb{H}(R) \oplus \mathbb{H}(R)\mathrm{t}$, and $\mathrm{t}^2 = -1$.

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Mutliplication in $\mathbb{O}(K)$: (Cayley-Dickson doubling process)
Let $a, b, c, d \in \mathbb{H}(K)$. Then $a + b\mathrm{t}$ and $c + d\mathrm{t} \in \mathbb{O}(K)$.

$$\forall\ a, b, c, d \in \mathbb{H}(K) \quad (a + b\mathrm{t})(c + d\mathrm{t}) = (ac + \lambda \overline{d} b) + (da + b\overline{c})\mathrm{t}.$$

Generalities on octonions II

Norm: non-degenerate quadratic form : $N(x) := x\bar{x}$ on $\mathbb{O}(R)$ that extends the one of $\mathbb{H}(R)$. With our settings, $N(\mathrm{i}) = N(\mathrm{j}) = N(\mathrm{t}) = 1$.

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Multiplicativity of the norm: $N(\alpha\beta) = N(\alpha)N(\beta)$

The unique factorization problem

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$\gcd(\alpha_0, \alpha_1, \alpha_2, \alpha_3) = 1$.
$N(\alpha) = p_1 \cdots p_s$ ($p_i \equiv 1 \bmod 4$, primes not necessarily disctinct).

Existence: There exists $\pi_i \in \mathbb{H}(\mathbb{Z})$, $N(\pi_i) = p_i$, such that:
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Uniqueness ? Impose that $\pi_{i,0} > 0$ and that $\pi_{i,0}$ is odd.
There exists a unique $\epsilon \in \mathbb{H}(\mathbb{Z})^\star = \{\pm 1, \pm \mathrm{i}, \pm \mathrm{j}, \pm \mathrm{ij}\}$,

$$\alpha = \epsilon \pi_1 \cdots \pi_s.$$

The sequence order $[\pi_1, \ldots, \pi_s]$ matters.

The unique factorization problem for octonions

1st step: Euclidean division: Given $\alpha, \beta \in \mathbb{O}(\mathbb{Z})$, $N(\alpha) > N(\beta)$, find $\gamma, \delta \in \mathbb{O}(\mathbb{Z})$ such that:

$$\alpha = \gamma\beta + \delta, \qquad N(\delta) < N(\beta).$$

Equivalently: Given $v \in \mathbb{Q}^8$, is there $w \in \mathbb{Z}^8$ such that $||v - w||_2 < 1$.

Not clear because $||(\frac{1}{2}, \cdots, \frac{1}{2})||_2 = \sqrt{2}$.

The unique factorization problem for octonions

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Does not work because $\mathbb{O}(\mathbb{Z})$ is not a maximal "order" (in analogy with algebraic integers: $\mathbb{Z}[\alpha] \subset \mathcal{O}_K$, where $K = \mathbb{Q}(\alpha)$).

Integral octonions

Characteristic equation: $\forall \alpha \in \mathbb{O}(K)$, holds:

$$X^2 - (\alpha + \bar{\alpha})X + N(\alpha) \equiv 0 \qquad \text{in} \qquad K[X]$$

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Coxeter, 1946 The integral octonions contains 7 distinct sub-algebras that are also maximal orders (lattices).

The 7 associative triads: Let k := ij. Each of the following 7 triplets generate a quaternion sub-algebra.

k, jt, it , j, it, kt , i, kt, jt , i, j, k , i, t, it , j, t, jt , k, t, kt

Coxeter algebra (E_8 lattice)

Coxeter's algebra $\mathcal{C}_{\mathbb{O}}$: This is one of the 7 maximal orders, associated to the associative triplet $\mathrm{i}, \mathrm{j}, \mathrm{k}$:

$$\mathrm{h} := \frac{1}{2}(\mathrm{i}+\mathrm{j}+\mathrm{k}+\mathrm{t}), \quad \mathcal{C}_{\mathbb{O}} := \mathbb{Z}+\mathrm{i}\mathbb{Z}+\mathrm{j}\mathbb{Z}+\mathrm{k}\mathbb{Z}+\mathrm{h}\mathbb{Z}+\mathrm{ih}\mathbb{Z}+\mathrm{jh}\mathbb{Z}+\mathrm{kh}\mathbb{Z}.$$

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Rehm (1993) Deduce a distortion of the Euclidean algorithm.
Existence of factorization.
Uniqueness of factorization: counting argument

Unique factorization: H. P. Rehm (1993)

Special case: $\alpha \in \mathbb{O}(\mathbb{Z})$, $N(\alpha) = p^k$, $p \equiv 1 \bmod 8$.

$$\alpha = \alpha_0 + \alpha_1 \mathrm{i} + \alpha_2 \mathrm{j} + \alpha_3 \mathrm{k} + \alpha_4 \mathrm{t} + \alpha_5 \mathrm{it} + \alpha_6 \mathrm{jt} + \alpha_7 \mathrm{kt}$$

α is primitive $\Leftrightarrow \gcd(\alpha_0, \ldots, \alpha_7) = 1$.

Existence: there are prime octonions $\pi_1, \ldots, \pi_k$, $N(\pi_i) = p$, such that:

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Uniqueness: Restrict the set of octonions of norm p to:

$$\mathscr{P}(p) := \{\alpha \in \mathbb{O}(\mathbb{Z}) \ : \ N(\alpha) = p \ , \alpha_0 \text{ is odd } \ , \ \alpha_0 > 0\}$$

There exists a unique sequence $[\mu_1, \ldots, \mu_k]$ in $\mathscr{P}(p)$ such that :

$$\alpha = \pm(\cdots(\mu_1\mu_2)\ldots)\mu_k \qquad (\mu_{i+1} \neq \overline{\mu_i})$$

$p^3 + 1$-regular infinite tree T_p

$\mathscr{C}ay(\langle \mathscr{P}(\mathrm{p}) \rangle \ , \ \mathscr{P}(\mathrm{p}))$ is the $p^3 + 1$-regular inifinite tree.

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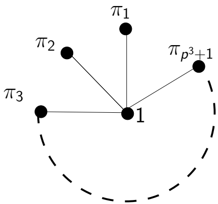

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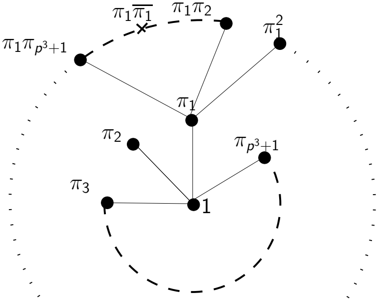

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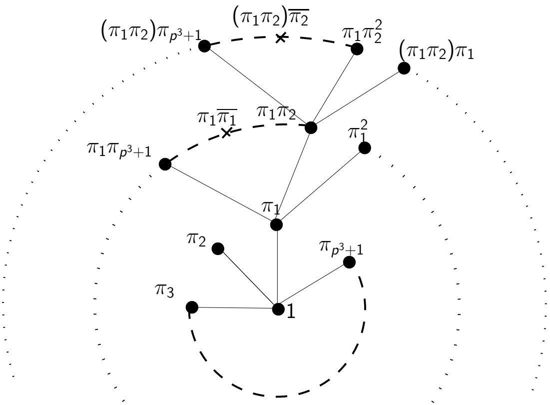

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Alternative algebra rules ... $(\alpha\beta)\bar{\beta} = \alpha(\beta\bar{\beta}) = \alpha N(\beta)$

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Walking on the tree: vertice $v \leftrightarrow \alpha = (\cdots(\pi_{i_1}\pi_{i_2})\ldots)\pi_{i_s}$, with $\pi_{i_\ell} \neq \overline{\pi_{i_\ell}}$.

Go forward (from the root) at v: right multiply α by $\pi \in \mathscr{P}(p) - \{\overline{\pi_{i_s}}\}$.

Go backward (from the root) at v: right multiply α by $\overline{\pi_{i_s}}$.

Finite regular quotients of the tree

$$\tau_q : \mathbb{O}(\mathbb{Z}) \to \mathbb{O}(\mathbb{F}_q)$$

Equivalence relation on the vertices: $v_1, v_2 \in V(T_p)$

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Theorem: The relation $\sim$ preserves the adjacency.

$\mathscr{X}_{p,q} := T_p/\sim$, finite p^3+1-regular quotient of T_p.

Algebraic interpretation in terms of Cayley graphs

$$\tau_q : \mathbb{O}(\mathbb{Z}) \to \mathbb{O}(\mathbb{F}_q) \qquad p \equiv 1 \bmod 8 \quad \text{and} \quad \left(\frac{p}{q}\right) = -1$$

Definition: Let

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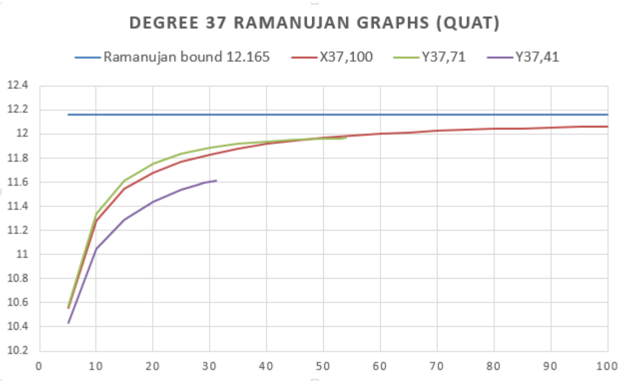
Let $\mathscr{S}(p,q) := \mu_q(\mathscr{P}(p))$, $\mathscr{X}_{p,q} = \mathscr{C}ay(\ \mathbb{O}(\mathbb{F}_q)^\star/\mathcal{Z}\ ,\ \mathscr{S}(p,q)\ )$

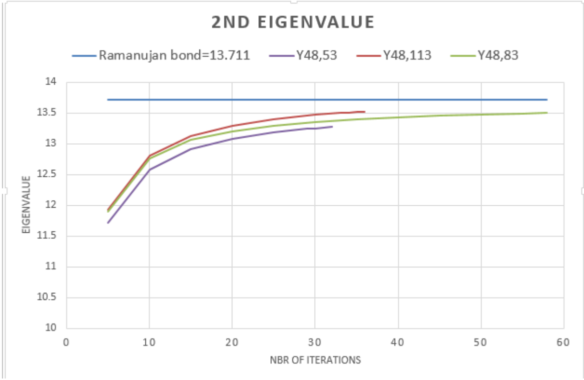
Some Numerical Experiments

- Implementation in Magma. $\leftarrow$ More than 2000 lines of codes.
- Computation of λ_1 the 2nd largest eigenvalue: Power Method.
- Computation of the girth: classical breadth-first search in the "mother" $p^3 + 1$-regular tree, until a "collision" is found when reducing mod q.

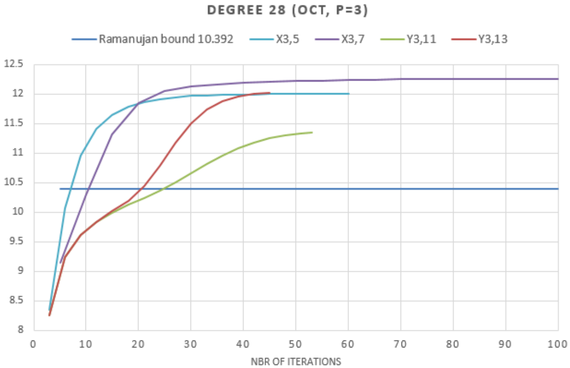
Results: 2nd eigenvalue for various degree 38 LPS graphs



Results: 2nd eigenvalue for various degree 48 LPS graphs



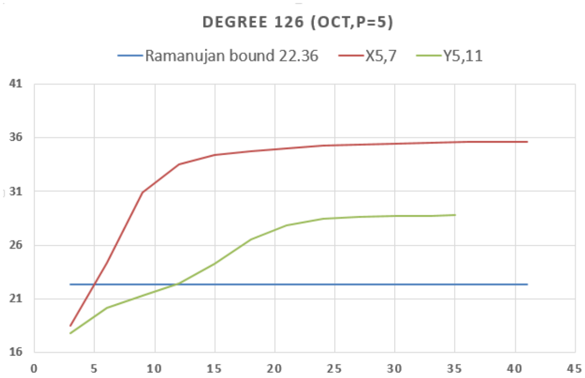
Results: 2nd eigenvalue for smallest degree 28 octo. graphs



31,373,160 vertices $Y_{3,13}$ required 24Go and 5h40 (one iteration 450s)

Failed for 410,333,760 vertices graph $X_{3,17}$ (after 30Go and 59hurs)

Results: 2nd eigenvalue for smallest degree 126 octo. graphs



$Y_{5,11}$ has 9,742,920 vertices. Required 11Go and 10hours (500s by iterations).

Implementation in MAGMA

Representation of Moufang loops $\mathbb{O}(\mathbb{F}_q)^\times/\mathcal{Z}$ (and of $\mathbb{H}(\mathbb{F}_q)^\star/\mathcal{Z}$)

- Construction of the doubling Cayley Dickson porocess $(\mathbb{R} \to \mathbb{C} \to \mathbb{H} \to \mathbb{O} \to \cdots)$ to generate automatically the multiplication tables on free modules of rank $2, 4, 8, \ldots$.

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- Use a "normal form" to represent quater/octo-nions in $\mathbb{H}(\mathbb{F}_q)^\times/\mathcal{Z}$ or $\mathbb{O}(\mathbb{F}_q)^\times/\mathcal{Z}$:

$$\mathbf{a} = (\alpha_0, \ldots, \alpha_7) \xrightarrow{\textit{Normal form}} \alpha_{\textit{first}}^{-1}\mathbf{a},$$

where $\alpha_{\textit{first}}$ is the first coordinate $\neq 0$.

Power method

Aim: Approximate largest eigenvalues of (symmetric) matrices.

$$\text{If } x_0 \notin E_{\lambda_0}, \qquad \lim_{\ell\to\infty} \frac{\|A^\ell x_0\|_2}{\|A^{\ell-1} x_0\|_2} = |\lambda_0|,$$

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- The product Ay can be done in case of Cayley graphs: $O(nd) = \tilde{O}(n)$ (if all elements are pre-computed and stored in an array).

Perspective

- Uneveness of the girth: *separate .pdf file*

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- MAGMA has some functionalities to compute automorphisms. Unlikley to work on large graphs, but is it possible to use these to guess at least some automorphism on smallest octonion graphs ?

Perspective

- Uneveness of the girth: *separate .pdf file*
- MAGMA has some functionalities to compute automorphisms. Unlikley to work on large graphs, but is it possible to use these to guess at least some automorphism on smallest octonion graphs ?
- THANK YOU FOR YOUR ATTENTION ! COMMENTS?

file:///C:/Program_Files_(x86)/Magma/htmlhelp/text1804.htm