Essential Mathematics for Global Leaders I

Spring 2019

Statistics

Lecture 3: 2019 May 13-20

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Where are we ? Today's plan

PART I. Notions of Probability 必要な確率論

1. Basic probability 確率論基礎

1.1 Counting

1.2 Probability rules

1.3 Conditional Probability, Independence, Bayes theorem

Lectures 1-2

2. Random variable 確率変数

2.1 Discrete random variable 離散確率変数

2.2 Some important distributions 幾つかの大事な分布

2.3 Operations on random variables 確率変数への作用

2.4 Expected value 期待値

2.5 Variance 分散

Chapter 2: Random variable

2.1 Discrete random variable

章2. 確率変数　2.1離散確率変数

- Experiment, sample space Ω, probability function $P(\,.\,)$.
  実験、標本空間、確率関数。
- A sample space Ω is discrete if it is finite, or “listable”
  $$\Omega = \{\omega_1, \omega_2, \omega_3, \dots\}$$
  Can be indexed by 1,2,3,...
  標本空間は有限または可付番ならば離散という。
- A random variable X on Ω is a function from the sample space Ω to the real numbers. $X: \Omega \to \mathbb{R}$
  標本空間Ωから実数全体$\mathbb{R}$への写像は確率変数という。
- “$X = a$” is the event $\{\omega \in \Omega \mid X(\omega) = a\}$

Example: bet on dice

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- Roll two dice, let (i, j) be the outcome.
  例：サイコロを二つ振って、結果を(i,j)と書く。
- Bet! 賭け!
  - win ＄700 if $i + j = 7$ （＄700が当たる）
  - loose ＄100 otherwise. （＄100損をする）
- Define the random variable X as follows:
  以下のように確率変数をXとする：
  - $X\big((i,j)\big) = 700$ if $i + j = 7$
  - $X\big((i,j)\big) = -100$ if $i + j \neq 7$

- "$X = 700$" is the event:
  $\{(1,6), (3,4), (4,3), (6,1), (5,2), (2,5)\}$.
- Thus $P(X = 700)$= 6/36=1/6.
- "$X = -100$"=$(X = 700)^c$ thus $P(X = -100) = 5/6$

Probability mass (density) function (pmf)
確率質量(密度)関数

- X a random variable on a discrete sample space Ω.
  離散標本空間Ω上の確率変数をXとする。
- Given $a \in \mathbb{R}$ let $p(a) \coloneqq P(X = a)$. This defines a function: $p: \mathbb{R} \to [0,1]$ called the probability mass function (pmf) of the discrete random variable X.
  関数$p: \mathbb{R} \to [0,1]$を定義し、離散確率変数Xの確率質量関数という。
- **Example**: rolling two dice, let (i, j) be the outcome.
  Random Variable *M:* $M(i,j) = \max(i,j)$.
- M is a discrete random variable and its pmf is

value	a:	1	2	3	4	5	6
pmf	$p(a)$:	$1/36$	$3/36$	$5/36$	$7/36$	$9/36$	$11/36$

Law of total probability: 総確率の法則: $\sum_{a=1}^{6} P(M = a) = 1$

Event defined by random variable's inequalities
確率変数の不等式による定義される事象

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- Let roll 2 dice and denote (i, j) the outcome.
  サイコロを二つ転がして結果を(i, j)と書く。
- $X(i, j) = i + j$ is a random variable.
- The probability mass function (確率質量関数) of X is:

a	2	3	4	5	6	7	8	9	10	11	12
$p(a)$	1/36	2/36	3/36	4/36	5/36	6/36	5/36	4/36	3/36	2/36	1/36

- **Definition:** '$X \leq t$' is the event $\{\omega \in \Omega \mid X(\omega) \leq t\}$.

- Example above:
- $X \leq 4 = \{(1,1), (1,2), (2,1), (1,3), (3,1), (2,2)\}$

Cumulative distribution function (cdf) 累積分布関数

- Let $X: \Omega \to \mathbb{R}$ be a random variable on a discrete sample space Ω. 離散標本空間Ω上の確率変数をXとする
- The pmf is defined as $p(a) = P(X = a)$ and the cdf is defined a $F(a) = P(X \leq a)$.
- **Example** (two dice rolls, $M(i, j) = \max(i, j)$).

$$p(a) = P(M = a), \qquad F(a) = P(M \leq a)$$

value	a:	1	2	3	4	5	6
pmf	$p(a)$:	1/36	3/36	5/36	7/36	9/36	11/36
cdf	$F(a)$:	1/36	4/36	9/36	16/36	25/36	36/36

- $F(0) = 0,\ F(1) = \frac{1}{36},\ F(1.5) = \frac{1}{36},\ F(5.1) = \frac{25}{36}$.

Cumulative distribution function (cdf) Question

- Let X be a random variable.

Values of $\boldsymbol{X}$ a	1	3	5	7
cdf $F(a)$	0.5	0.75	0.9	1

1. What is $P(X \leq 3)$?
   a) .25 b) .5 c) .75 d) 1.25

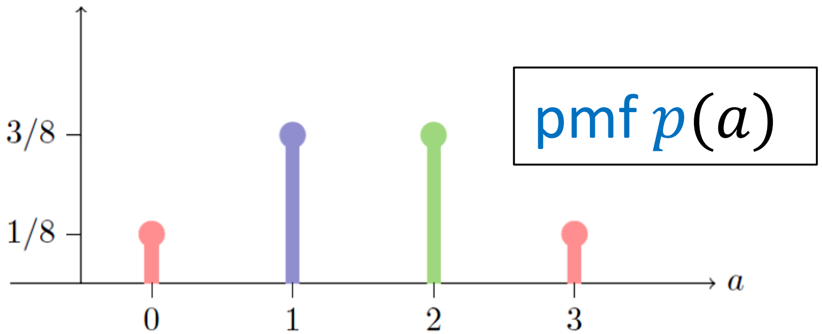
2. What is $P(X = 3)$?
   a) .25 b) .5 c) .75 d) 1.25

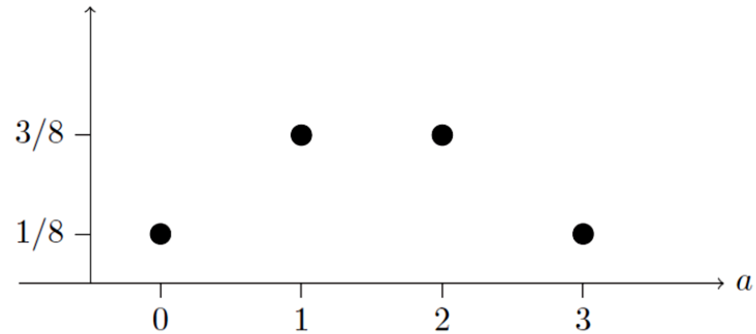
- 1. **Answer**: (c) 0.75 $P(X \leq 3) = F(3) = \ .075$
- 2. **Answer**: (a) 0.25 $P(X = 3) = .75 - \ .5 = .25$

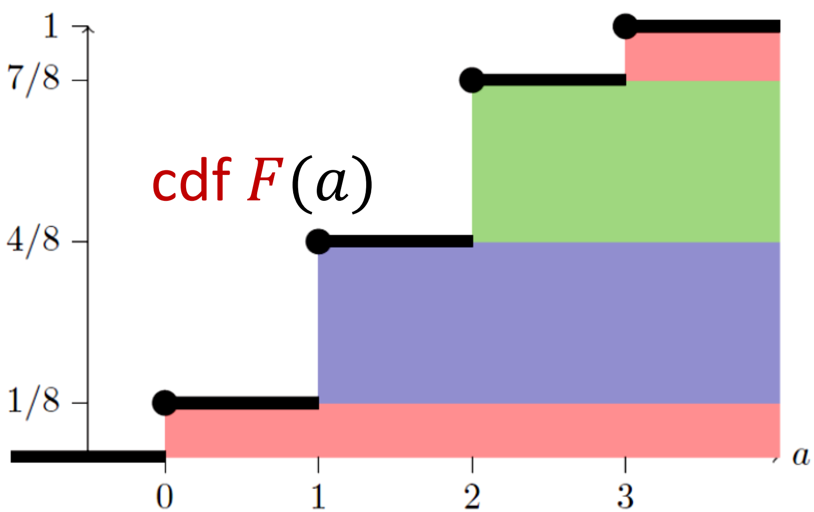
Graphs of pmf and cdf
(cumulative distribution function 累積分布関数)

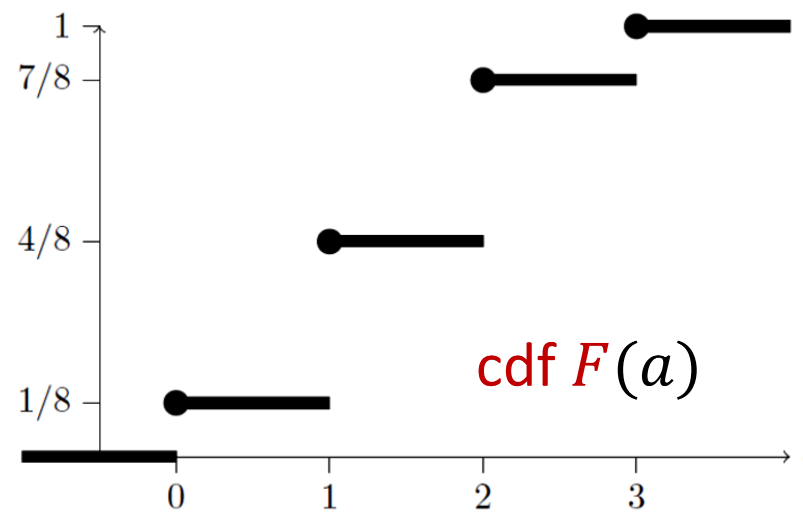
- Experiment: 3 tosses of a coin.
- X: number of heads. H の回数
- $p(a) = P(X = a),\ F(a) = P(X \le a)$

Value of $\boldsymbol{a}$	0	1	2	3
pmf $p(a)$	1/8	3/8	3/8	1/8
cdf $F(a)$	1/8	4/8	7/8	1





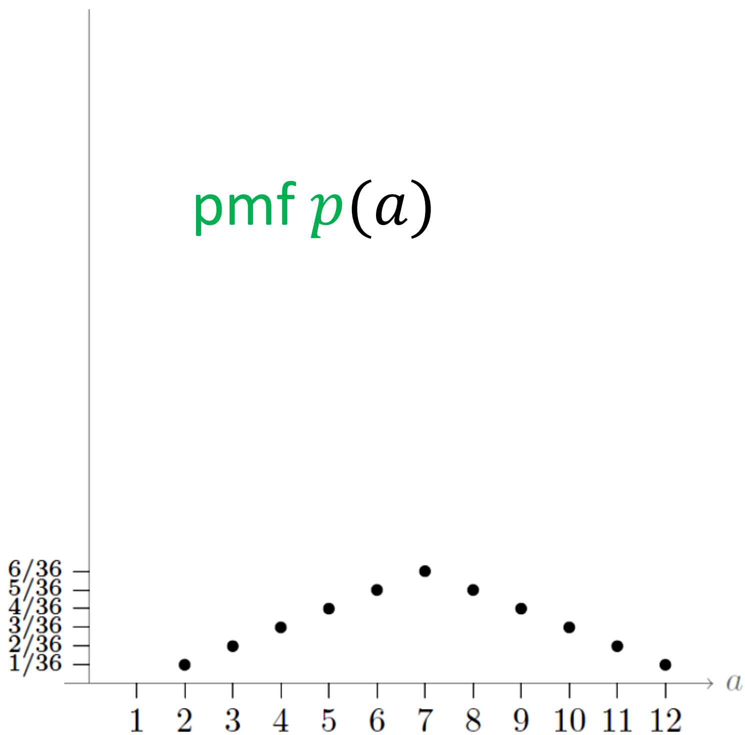


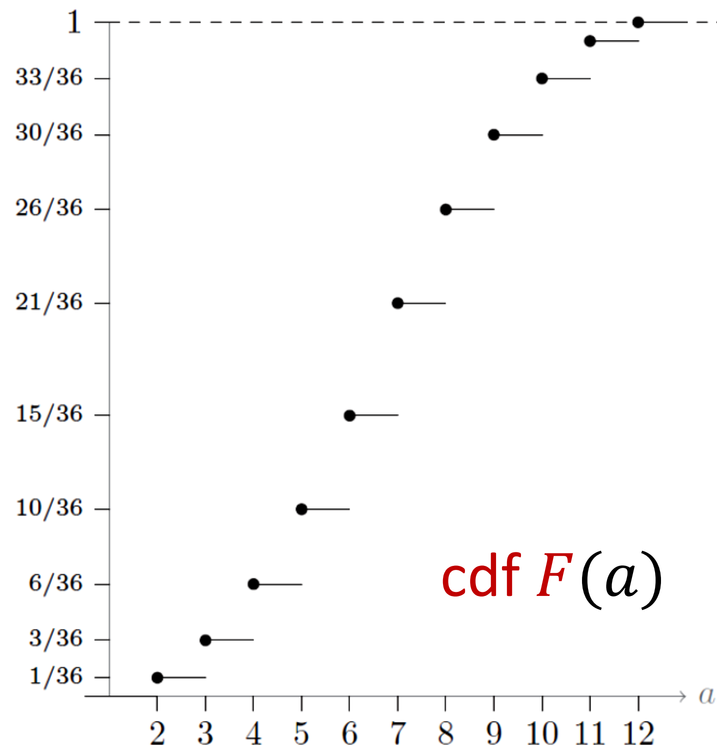


pmf and cdf of another rv (random variable)
また別の確率変数の確率質量関数と累積分布関数

Roll two dice, (i, j) an outcome, $X = i + j$.

a	2	3	4	5	6	7	8	9	10	11	12
$p(a)$	1/36	2/36	3/36	4/36	5/36	6/36	5/36	4/36	3/36	2/36	1/36





Ch.2 Random variables

2.2 Some important distributions

- n要素上の一様確率分布　Uniform(n)
  $\Omega = \{1,2,\ldots,n\}$ same probability for all i: $P(X=i) = 1/n$

- 成功率pのベルヌーイ分布。
- Bernoulli(p) $=$ 1 (success) with probability p
  0 (failure) with probability $1-p$

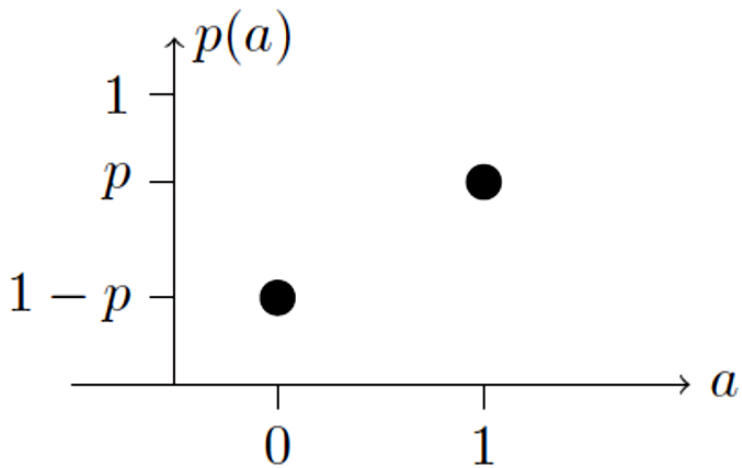
  (coin toss) $=$ 1 (HEADS) with probability p
  0 (TAILS) with probability $1-p$

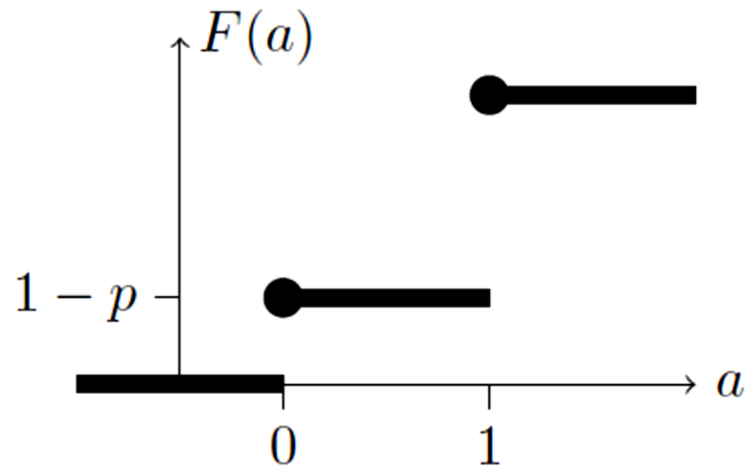
- 二項分布
- Binomial(n,p) $=$ Nbr. of successes in n independent Bernoulli(p) trials.
  $=$ Nbr. of HEADS in n tosses of a coin.

- 幾何分布
- Geometric(p) $=$ Nbr of TAILS before 1^{st} HEADS in a sequence of independent Bernoulli(p) trials.

1st example: Bernoulli(p)　ベルヌーイ分布

- $X \sim Bernoulli(p)$
  Read: “X follows a Bernoulli distribution with parameter p”
  確率変数Xは成功率pのベルヌーイ分布に従う。
  - Experiments with 2 outcomes: Success=1 or Failure=0
  - Sample space is divided into: $\Omega = \{X = 0\} \cup \{X = 1\}$.
  - Probability function: $P(X = 0) = p,\ \ P(X = 1) = 1 - p$。





2^{nd} example: Binomial(n,p) 二項分布

- $X \sim Binomial(n, p)$
  - Parameters（母数）: n nbr. of independent Bernoulli trials with parameter p.
    成功率pのベルヌーイの独立の試行回数n
  - '$X = k$' ☛ 'k successes in n independent trials'
    成功率pのベルヌーイ試行を独立にn回行ったときの成功回数がkである事象を表す。

- **Example**: 5 tosses of a coin (= 5 Bernoulli(p)) $P(H) = p$.
  - Sample space: $\{HHHHH, HHHHT, \ldots, TTTTH, TTTTT\}$
  - '$X = 2$' $= \{HHTTT,\ HTHTT,\ HTTHT,\ HTTTH,\ THHTT,\ THTHT,\ THTTH,\ TTHHT,\ TTHTH,\ TTTHH\}$
  - All of these 10 outcomes have the same probability:
    $P(\omega) = p^2(1-p)^3$, for any $\omega \in$ '$X = 2$'
  - Thus: $p(2) = P(X = 2) = 10p^2(1-p)^3$

Pmf of a Binomial(n,p) 二項分布の確率質量関数

- More generally: $X \sim Binomial(n,p)$
- $P(X=k) =$ (number of all combinations of k successes out of n trials)$\times p^k(1-p)^{n-k}$.

$$p(k) = P(X=k) = \binom{n}{k} p^k(1-p)^{n-k}$$

- Total law of probability: (総確率の法則 Lecture2, p.17)

$$\sum_{k=0}^{n} p(k) = \sum_{k=0}^{n} \binom{n}{k} p^k(1-p)^{n-k} = 1\,.$$

- Why this is true? ☛ Binomial Expansion 二項展開
- $1 = (p+1-p)^n = (p+q)^n = \sum_{k=0}^{n}\binom{n}{k}p^k q^{n-k}$

Binomial's pmf and cdf (cumulative distri-bution function 累積分布関数)

- $Binomial\left(20, \frac{1}{2}\right)$
- $Binomial(20, 0.7)$
- $Binomial\left(40, \frac{1}{2}\right)$

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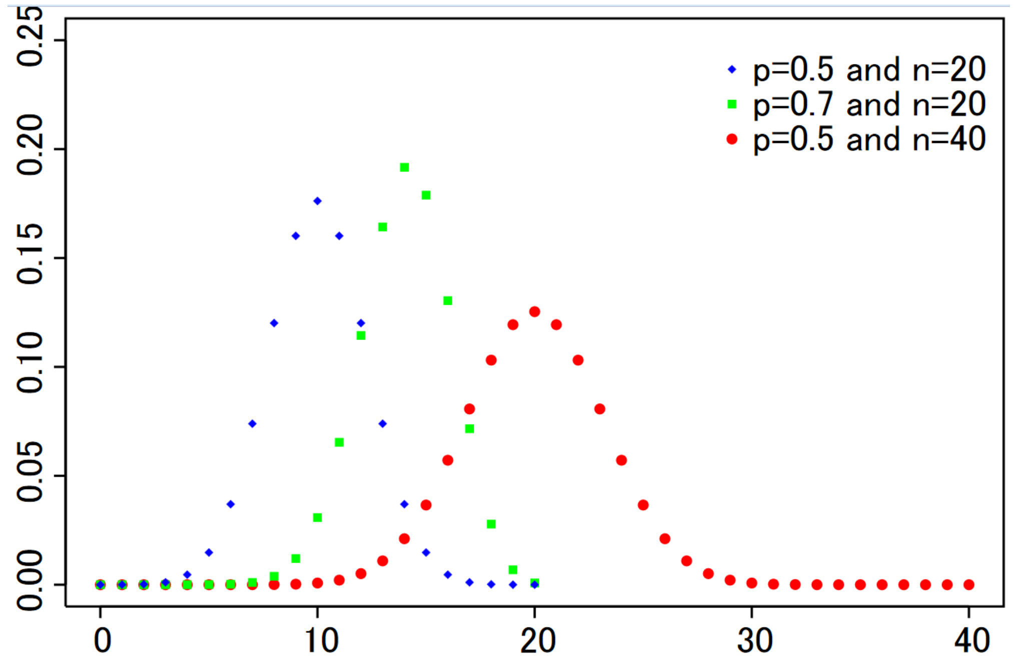
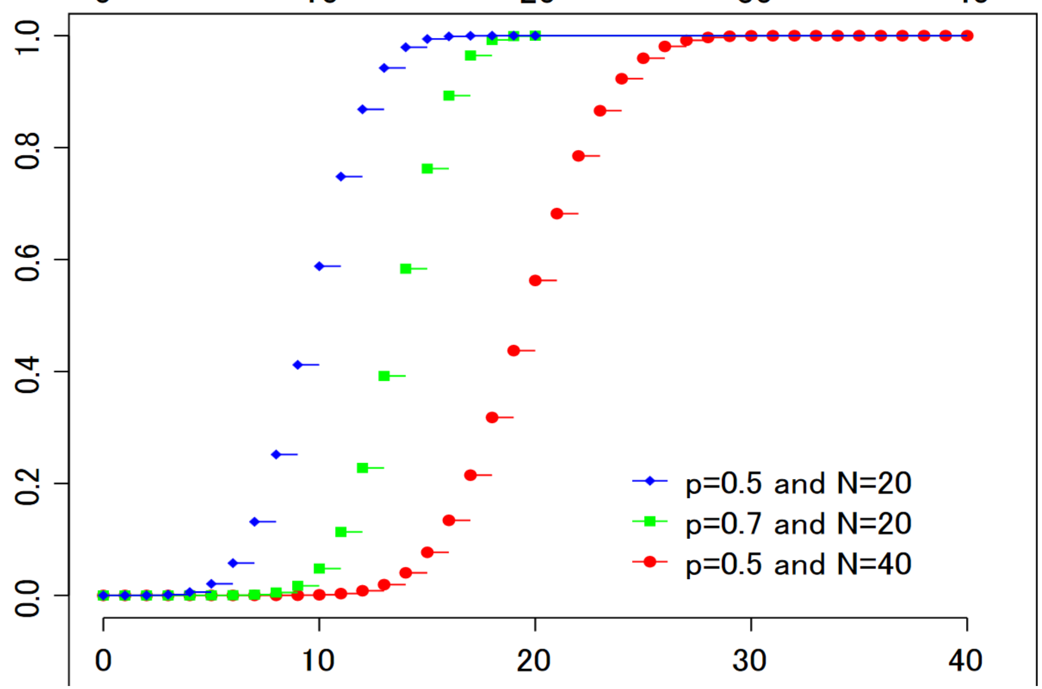
- This "bell-shaped" is characteristic (discrete version of Gaussian distributions)

このベル・カーブ（鐘形曲線）は特徴で、正規分布の離散化した分布と言ってもいい。





3rd example: Geometric(p) 幾何分布

- $X \sim Geometric(p)$.
  - Sample space: infinite independent Bernoulli trials with success probability p
    無限の独立の成功率pのベルヌーイ試行
  - Probability mass function (pmf):
    $P(X = k) = P($k-1 first trials fail, k-th trial succeeds)
    P(最初のk-1回の試行は失敗、第k回目は成功)

$$P(X = k) = (1 - p)^{k-1}p.$$

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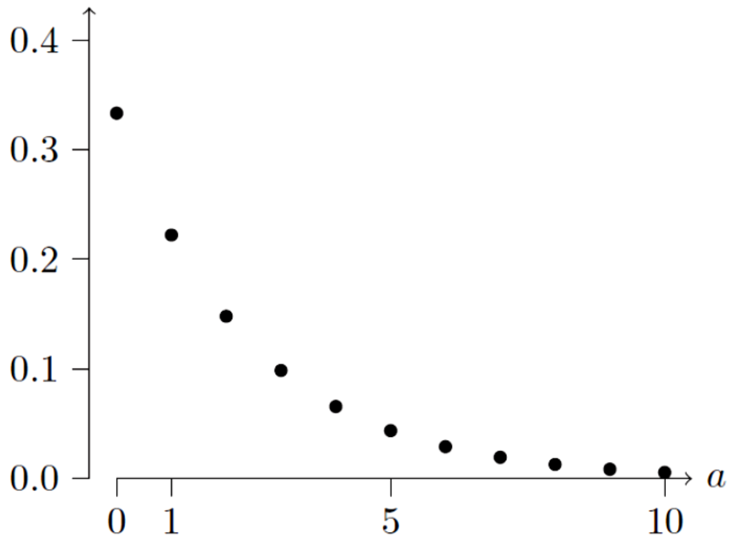
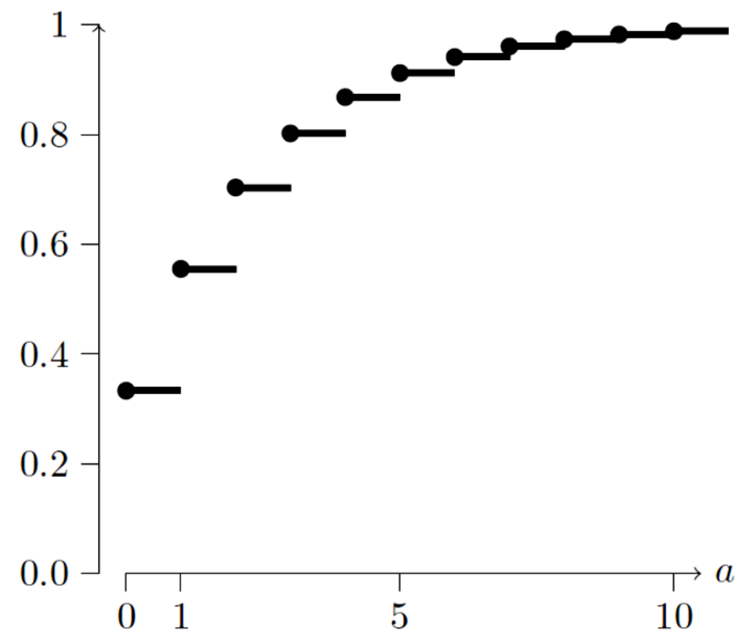
- **Example**: soccer player goals 9/10 penalty kicks in average.
  サッカー選手は9/10でペナルティキックを決める。
  Let X be the number of times she succeeds before her first fail. 初めてゴール失敗する前にゴールの決めた回数。
  $X \sim Geometric(1/10)$

Shot a fails	1	2	3	4	5	6	7	8	9	10
$p(a)$	.1	.09	.081	.073	.066	.06	.053	.048	.043	.039

Pmf and cdf of a Geometric(p)
p幾何分布の確率質量関数と累積分布関数





Probability mass function (left) and cumulative distribution function (right) graphs of a $Geometric\left(\frac{1}{3}\right)$.
$Geometric\left(\frac{1}{3}\right)$の確率質量関数(左)と累積分布関数(右)。

2.3 Operations on random variables

- $X, Y: \Omega \to \mathbb{R}$ two random variables (確率変数の2つ)
- $X(\omega)$ and $Y(\omega)$ are just numbers ! ただの実数だ！
- **Multiplication (掛け算)**
  $X(\omega) \cdot Y(\omega),\ X(\omega)^2,\ \sqrt{X(\omega)}$ etc. are other numbers.
  ☞ $XY, X^2, \sqrt{X}$ etc. are random variables.
- **Addition: (足し算)**
  $X(\omega) + Y(\omega)$ is another number…
  ☞ $X + Y$ is also a random variable
- **Division (割り算)**
  If $Z(\omega) \neq 0$ then $X(\omega)/Z(\omega)$ is a number
- **Example**: $\frac{3X(\omega)}{Z(\omega)} + Y(\omega)^2$ is also a number.

  ☞We can define the random variable $V = \frac{3X}{Z} + Y^2$.

Sum of ind. Bernoulli(p) = Binomial(. ,p)

- Experiments: 5 tosses of a coin.　コイン投げ5回
- Assumption: $P(H) = p$. All toss are independent.
  仮定：表の確率はp.　コイン投げは互いに独立。
- Sample space: $\Omega = \{HHHHH, HHHHT, \dots, TTTTT\}$
- X_i: result of the i-th toss (0=T, 1=H)　第i回目の結果
- Y: number of H (表の回数)

$$Y = X_1 + X_2 + X_3 + X_4 + X_5$$

- For example, if $\omega = HTHHT$ then
  - $X_1(\omega) = X_3(\omega) = X_4(\omega) = 1$ and
  - $X_2(\omega) = X_5(\omega) = 0$.
  - Therefore $Y(\omega) = 3$.
- What is the distribution of Y:　$Y \sim Binomial(5, p)$
- What is the distribution of X_i:　$X_i \sim Bernoulli(p)$

Sum of two independent Binomial(. ,p)
独立の成功率pのベルヌーイ確率変数二つの和

- If $X \sim Binomial(10, p)$ and $Y \sim Binomial(7, p)$.
- Assume that X and Y are independent.
  X とYは独立な二項分布に従う確率変数。
- What is the distribution of $X + Y$?　$X + Y$の分布は？

- **Answer**: Previous slide☞ $X = X_1 + X_2 + \cdots + X_{10}$
- And ☞ $Y = Y_1 + Y_2 + \cdots + Y_7$
- $Y_j, X_i \sim Bernoulli(p)$. Mutually independent
  互いに独立である。
- $X + Y = X_1 + \cdots + X_{10} + Y_1 + \cdots Y_7$ sum of 17 independent p-Bernoulli variables.
  $X + Y$は互いに独立の17つのpベルヌーイ分布和の和：
- Previous slide $X + Y \sim Binomial(17, p)$.
- 注意 ：If $X \sim Binomial(n, p)$ and $Y \sim Binomial(n, q)$ then $X + Y$ is not a binomial(n,p+q)

Ch.2 Random variables

2.4 Expected value 期待値

- Roll a 6-sided dice. What is the average value?

a) 3 b) 3+1/3 c) 3.5 d) 3 + 2/3 e) 4

Bet ! 賭け！ (slide 4)

- Two 6-faces dice are rolled. Result : i, j
  6面サイコロを二つ振る。結果: i, j
- Win 700\$ if $i + j = 7$,
  otherwise loose 100\$.

☛ How much do you expect to win on average per trial?
試行ごとに平均的にどれくらいお金が当たるのを期待するか。例えば、試行数をNをとし、確率変数$X(i,j) =$ 700 ($i + j = 7$) or -100 ($i + j \neq 7$)を使う。

- Answer: $\frac{1}{N}\left(P(X = 700) \cdot 700N - P(X = -100) \cdot 100N \right) = \frac{100}{3}$

Expected value. Definition 1

- Given a discrete random variable $X: \Omega \to \mathbb{R}$ from the sample space Ω, its expected value is:

$$E(X) = \sum_{\omega \in \Omega} X(\omega) \cdot P(\omega)$$

- In the previous page example: sample space is $\Omega = \{(1,1), (2,1), (3,1), \dots, (6,1), (6,2), \dots, (6,6)\}$.
- $X\big((i,j)\big) = 700$ if $i + j = 7$,
  and $X\big((i,j)\big) = -100$ if $i + j \neq 7$.
- $E(X) = 700\left(\sum_{i+j=7} \frac{1}{36}\right) - 100\left(\sum_{i+j\neq 7} \frac{1}{36}\right)$
  $= 700 \cdot \frac{6}{36} - 100 \cdot \frac{30}{36} = \frac{700-500}{6} = \frac{200}{6} = \frac{100}{3}$.

Expected value. Definition 2

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- If X is a random variable that takes the value $x_1, x_2, \ldots, x_n$
  確率変数Xは値$x_1, x_2, \ldots, x_n$をとるときに、

$$E(X) = \sum_{i=1}^{n} x_i \cdot p(x_i) = \sum_{i=1}^{n} x_i P(X = x_i).$$

- Example: $X \sim Bernoulli(p)$. What is $E(X)$?
- $E(X) = 1 \cdot p + 0 \cdot (1-p) = p$, $P(X=1).1 + P(X=0).0$

- Sum of two dice: $Y(i,j) = i + j$ (slide 6). What is $E(Y)$?

a	2	3	4	5	6	7	8	9	10	11	12
$p(a)$	1/36	2/36	3/36	4/36	5/36	6/36	5/36	4/36	3/36	2/36	1/36

- $E(Y) = \frac{1}{36}(2+6+12+20+30+42+40+36+30+22+12)$
- $E(Y) = 252/36 = 7.$

Interpreting expected value (期待値の解釈)

a) Would you accept a gamble that offers a 10% chance to win ＄95 and a 90% chance of losing ＄5?
＄5の損をする確率90%と＄95が当たる確率10%の賭博に賭けるか。

b) Would you pay ＄5 to participate in a lottery that offers a 10% percent chance to win ＄100 and a 90% chance to win nothing?
＄100が当たる確率10%と何も当たらない確率90%の宝くじに参加するように＄5を払うか。

- Find the expected value of your change in assets in each case.
いずれの場合にも資産増減の期待値を求めよ。
- **Answer:** Both a) and b) are same: $E(X) = 5$

Computation of $E(X)$

- We have: $E\ (X) = \sum_i x_i\, p(x_i)$

- Given any numbers a and b: 各数a, bに対して：

1. $E(aX + b) = \ aE(X) + b$

2. $E(X + Y) = \ E(X) + E(Y)$ (*even when X and Y are dependent. X とY は独立でなくても成り立つ*).

- Given a function h (for example $h(x) = x^2$)

3. $E\big(h(X)\big) \ = \sum_i h(x_i)\, p(x_i)$ (=$\sum x_i^2 p(x_i)$)

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Example1. $E\big(Binomial(n,p)\big) = np$

- $X \sim Binomial(n,p)$.
- Find $E(X)$ (remember that $X = X_1 + X_2 + \cdots + X_n$ where $X_i \sim Bernoulli(p)$).

- **Answer:** Slide 23 ☞ $E(X_i) = p$
- Previous slide, Property 2:

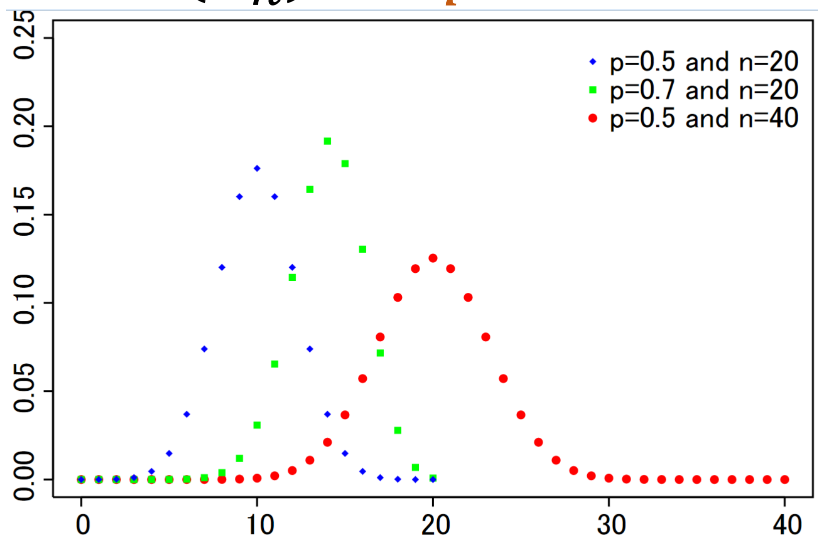
$$E(X) = E(X_1) + \cdots + E(X_n) = np$$

Example: (Slide 14)

$Binomial(20,0.5)$ $E(X) = 10$

$Binomial(20,0.7)$ $E(X) = 14$

$Binomial(40,0.5)$ $E(X) = 20$



Example2. Change seat.
Report Exercise 3

- Suppose that people sit around a table, got up, turn around and sat down again randomly.
  テーブルを囲んで座る人が、立っ回ってランダムに椅子に座る。
- What is the expected value of the number of people sitting in their original seat?
  元の椅子に座った人数の期待値は何か？
- *Hint: Define X_i the random variable as $X_i = 1$ to be the event 'i-th person has the same seat' and $X_i = 0$ to be the event 'i-th person has changed seat'. What is the distribution of the X_i?*
  'i人目は同じ椅子に座る'を$X_i = 1$とし、'i人目は別の積に座る'を$X_i = 0$とする確率変数を定義せよ。。。

Practice computation 計算練習. $E(X^2)$

- Roll a 6-sided dice and let X be the value. Let $Y = X^2$.

X	1	2	3	4	5	6
Y	1	4	9	16	25	36
prob	1/6	1/6	1/6	1/6	1/6	1/6

- Compute $E(Y)$.
- $E(Y) = E(X^2) = \frac{1}{6} \cdot 1 + \frac{1}{6} \cdot 4 + \frac{1}{6} \cdot 9 + \frac{1}{6} \cdot 16 + \frac{1}{6} \cdot 25 + \frac{1}{6} \cdot 36 = 15.167$.

Expected value of Geometric(p) 幾何分布の期待値

- If $X \sim Geometric(p)$ then
$$E(X) = \sum_{k=1}^{\infty} (1-p)^{k-1} p$$
- It's a mathematician's job to compute that…
このような計算は数学者の仕事である。。
- We find: $$E(X) = \frac{1}{p} - 1.$$
- **Example**: (slide 16)
Soccer player goals 9/10 of its penalty kicks. In average how many times she will goal before she misses one for the first time?
サッカー選手は10分の９のペナルティキックを決める。平均的に、初めてショットを失敗する前に何回ゴールを決めるか？
- **Answer:** Let X be the number of times she goals before she fails. Then $X \sim Geometric(1/10)$. Thus $E(X) = 10 - 1 = 9$.

Chapter 2: Random variable

2.5 Variance of discrete random variables

章2. 確率変数　2.5　離散確率変数の分散

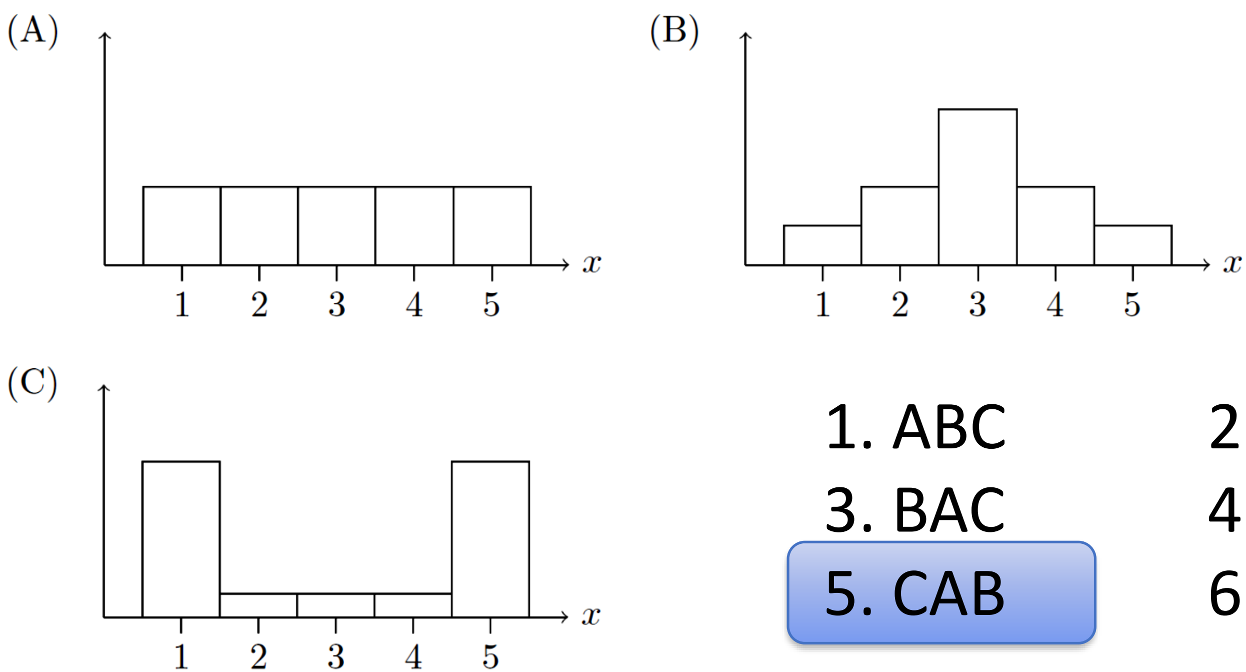
- X a discrete random variable with mean $E(X) = \mu$.
  X を期待値=平均値$E(X) = \mu$の離散確率変数とする。
- Meaning: spread of probability mass about the mean.
  意味：確率質量の平均値あたりのまき散らす傾向。
- Definition as expectation of the random variable $(X - \mu)^2$
  $Var(X) = E\left((X - \mu)^2\right) = E(X^2) - 2E(X)\mu + \mu^2 = E(X^2) - \mu^2$
- Computation as sum:

$$Var(X) = \sum_{i=1}^{n} p(x_i)(x_i - \mu)^2$$

- Standard deviation　標準偏差 $\sigma = \sqrt{Var(X)}$.

Question (understanding concept)

The graphs below give the pmf for 3 random variables.
以下のグラフは確率変数の確率質量関数を三つ表す。
Order them by size of standard deviation from biggest to smallest.　減少する順に標準偏差を分類せよ。



1. ABC
2. ACB
3. BAC
4. BCA
5. CAB
6. CBA

Example: computation from a table (I)

- Compute the variance and standard deviation of X.

values x	1	2	3	4	5
pmf $p(x)$	$1/10$	$2/10$	$4/10$	$2/10$	$1/10$

- Start by computing the mean (=expected value):

$$\mu = \frac{1}{10} + \frac{4}{10} + \frac{12}{10} + \frac{8}{10} + \frac{5}{10} = 3$$

- Then add a line to the table for $(X - \mu)^2$.

values X	1	2	3	4	5
pmf $p(x)$	$1/10$	$2/10$	$4/10$	$2/10$	$1/10$
$(X-\mu)^2$	4	1	0	1	4

- Compute $Var(X) = E\left((X - \mu)^2\right)$ and $\sigma = \sqrt{Var(X)}$.

$\frac{1}{10} \cdot 4 + \frac{2}{10} \cdot 1 + \frac{4}{10} \cdot 0 + \frac{2}{10} \cdot 1 + \frac{1}{10} \cdot 4 = 1.2, \quad \sigma = \sqrt{1.2}.$

Example: computation from a table (II)

- Compute the variance and standard deviation of X.

values x	1	2	3	4	5
pmf $p(x)$	$1/10$	$2/10$	$4/10$	$2/10$	$1/10$

- From the previous slide: $\mu = 3$.
- We can use the other formula for the variance:

$$Var(X) = E(X^2) - \mu^2$$

- $E(X^2) = 1 \cdot \frac{1}{10} + 2^2 \cdot \frac{2}{10} + 3^2 \cdot \frac{4}{10} + 4^2 \cdot \frac{2}{10} + 5^2 \cdot \frac{1}{10}$
- $E(X^2) = \frac{1}{10} + \frac{8}{10} + \frac{36}{10} + \frac{32}{10} + \frac{25}{10} = \frac{102}{10} = 10.2$

- $Var(X) = E(X^2) - \mu^2 = 10.2 - 9 = 1.2$.

Question

- True or False. If $Var(X) = 0$ then X is constant.
  もし$Var(X) = 0$ならば、Xは不変である。

1. True

2. False

- **True**: $Var(X) = \sum_{i=1}^{n} p(x_i)(x_i - \mu)^2 = 0$.
- For all i, $p(x_i)(x_i - \mu)^2 \geq 0$.
- Therefore either $p(x_i) = 0$ and the event $X = x_i$ never occurs, ...
- .....or $p(x_i) > 0$ and $x_i - \mu = 0$.
- Thus, $x_i = \mu$ and $X = \mu$ (is constant).

Operation on variances 分散への作用

- If a and b are any numbers then 任意の数aとbに対して
  $Var(a\,X + b) = a^2 Var(X)$, $\sigma_{aX+b} = |a| \cdot \sigma_X$

- If X and Y are independent random variables then
  X と Yを独立な確率変数とすると：
  $$Var(X + Y) = Var(X) + Var(Y).$$

- **Proof (証明)**: follows from the same properties for the expectation.
  証明：期待値の同様な性質から得られる様式である。
  (☛slide 25)

Variance of Bernoulli, Binomial, Geometric

1. $X \sim Bernoulli(p)$ then $Var(X) = p(1-p)$.

   Proof (証明) $E(X^2) = 0^2 \cdot (1-p) + 1^2 \cdot p = p$
   $Var(X) = E(X^2) - E(X)^2 = p - p^2 = p(1-p)$

2. $X \sim binomial(n, p)$ then $Var(X) = np(1-p)$.

   Proof (証明) $X = X_1 + \cdots + X_n$,
   $X_i \sim Bernoulli(p)$ independent
   $Var(X) = Var(X_1) + \cdots + Var(X_n) = nVar(X_i) = np(1-p)$

3. $X \sim Geometric(p)$ then $Var(X) = \frac{1-p}{p^2}$ (no proof)

4. $X \sim Uniform(n)$ then $Var(X) = E(X^2) - E(X)^2 = \frac{n^2-1}{12}$

$$E(X^2) = \sum_{i=1}^{n} i^2 P(X = i) = \frac{1}{n}\left(\sum_{i=1}^{n} i^2\right) = \frac{1}{n}\frac{n(n+1)(2n+1)}{6}, \qquad E(X)^2 = \frac{n^2+2n+1}{4}$$

Review table of distribution まとめ

分布 Distribution	値域 range X	確率質量関数 pmf $p(x)$	期待値 mean $E(X)$	分散 variance $\mathrm{Var}(X)$
Bernoulli(p)	0, 1	$p(0) = 1 - p, \quad p(1) = p$	p	$p(1-p)$
Binomial(n, p)	$0, 1, \ldots, n$	$p(k) = \binom{n}{k} p^k (1-p)^{n-k}$	np	$np(1-p)$
Uniform(n)	$1, 2, \ldots, n$	$p(k) = \frac{1}{n}$	$\frac{n+1}{2}$	$\frac{n^2-1}{12}$
Geometric(p)	$0, 1, 2, \ldots$	$p(k) = p(1-p)^k$	$\frac{1-p}{p}$	$\frac{1-p}{p^2}$

Review table: formula for expectation and variance
期待値と分散の公式のまとめ表

Expected Value:	期待値	**Variance:** 分散
Synonyms: 類語	mean, average 平均値	
Notation: 記号	$E(X)$, μ	$\mathrm{Var}(X)$, σ^2
Definition: 定義	$E(X) = \sum_j p(x_j) x_j$	$E((X-\mu)^2) = \sum_j p(x_j)(x_j-\mu)^2$
Scale and shift:	$E(aX+b) = aE(X)+b$	$\mathrm{Var}(aX+b) = a^2\mathrm{Var}(X)$
Linearity: 線形性	(for any X, Y) $E(X+Y) = E(X)+E(Y)$	(for X, Y independent) $\mathrm{Var}(X+Y) = \mathrm{Var}(X)+\mathrm{Var}(Y)$
Functions of X:	$E(h(X)) = \sum p(x_j)\, h(x_j)$	
Alternative formula:	別公式	$\mathrm{Var}(X) = E(X^2) - E(X)^2 = E(X^2) - \mu^2$

Exercise1 **(Report Exercise 6)**

- $E(X+Y)=E(X)+E(Y)$ is true even if X and Y are not independent. Why do we need independence of X and Y for $Var(X+Y)=Var(X)+Var(Y)$ to be true ?
  XとYは独立でなくても$E(X+Y)=E(X)+E(Y)$が成り立つ。一方、$Var(X+Y)=Var(X)+Var(Y)$が正しくするように、XとYの独立性が必要の理由はなにか？

Exercise 2 (TO DO in Class)

- A random variable X takes values -1, 0, 1 with probabilities 1/8, 2/8, 5/8 respectively.
  - (a) Compute E(X).
  - (b) compute E(X^2)
  - (c) Compute Var(X).