Chapter 3 連立1次方程式解法:反復法

Section 3: Overview of Relaxation method and conjugate gradient

(過剰)加速緩和法　と共役勾配法　(概要)

- 加速緩和法とは、基本的にGauss-Seidel反復法に基づいて加速する目的とする修正を含む方法である。
  Relaxation methods are based on Gauss-Seidel iteration with modifications to accelerate the convergence.

- 共役勾配法とは、降下法型の反復計算方である。
  Aは（対称）正定値だと仮定する。
  The conjugate gradient method is an iterative descent method. (Assume that A is symmetric definite positive)

$$Ax^\star = b \iff x^\star = \min_{x \in \mathbb{R}^n} x^T A x - 2x^T b$$

Subsection 1: Relaxation method 緩和法
Residual error 残差

- 残差 “residual error”

$$\boldsymbol{r}_{\boldsymbol{i}}^{(k)} = \boldsymbol{b} - A\boldsymbol{x}_i^{(k)}$$

- Gauss-Seidelのアルゴリズムによる,

$$\boldsymbol{x}_1^{(1)} = \left(x_1^{(0)}, \dots, x_n^{(0)}\right)^T = \boldsymbol{x}^{(0)}$$
$$\boldsymbol{x}_2^{(1)} = \left(x_1^{(1)}, x_2^{(0)}, \dots, x_n^{(0)}\right)^T$$
$$\boldsymbol{x}_i^{(1)} = \left(x_1^{(1)}, x_2^{(1)}, \dots, x_{i-1}^{(1)}, x_i^{(0)}, \dots, x_n^{(0)}\right)^T$$
$$\vdots$$
$$\boldsymbol{x}_1^{(2)} = \left(x_1^{(1)}, x_2^{(1)}, \dots, x_n^{(1)}\right)^T = \boldsymbol{x}^{(1)}$$

- $\boldsymbol{r}_i^{(k)} = \left(r_{1,i}^{(k)}, r_{2,i}^{(k)}, \dots, r_{i-1,i}^{(k)}, r_{ii}^{(k)}, \dots, r_{ni}^{(k)}\right)^T$ と書く。

- この公式　$x_i^{(k)} = x_i^{(k-1)} + \frac{r_{ii}^{(k)}}{a_{ii}}$ をGauss-Seidelの定義に従う。

Toward a smaller residue 残差をより地作する

- **Principle 原理**: Choose $x_i^{(k)}$ at each step make $\left\| \boldsymbol{r}_{i+1}^{(k)} \right\|$ smaller:
- 加速係数ωを$x_i^{(k)} = x_i^{(k-1)} + \frac{r_{ii}^{(k)}}{a_{ii}}$入れて

$$x_i^{(k)} = x_i^{(k-1)} + \omega \frac{r_{ii}^{(k)}}{a_{ii}}$$

- 行列の形：$T_\omega = (D - \omega L)^{-1}[(1-\omega)D + \omega U]$

$$c_\omega = \omega(D - \omega L)^{-1} b$$

- $$x^{(k+1)} = T_\omega x^{(k)} + c_\omega b$$

- $\omega = 1$のとき、

$$x^{(k+1)} = (D-L)^{-1}U\,x^{(k)} + (D-L)^{-1}b$$

つまり、Gauss-Seidel反復である。

- $\rho(T_\omega)$ is difficult to compute in general

Choice of ω

- If $\omega > 1$ then such methods are called over-relaxation.
  $\omega > 1$ の時、過剰(加速)緩和法という。

  SOR: Successive Over-Relaxation
  逐次過剰緩和法

- Used in lot for linear systems coming from discretization of EDP (tridiagonal, diagonal dominant)
  偏微分方程式を差分近似して与える連立1次方程式に有用である。(対角優位行列、三重対角行列)

Example

- Consider the following tridiagonal system:

$$\begin{aligned} 4x_1 + 3x_2 &= 24 \\ 3x_1 + 4x_2 - x_3 &= 30 \\ - x_2 + 4x_3 &= -24 \end{aligned}$$

- Compute the matrices D, L, U and $T_g = (D - L)^{-1}U$
- Compute the vector $c_g = (D - L)^{-1}b$
- Deduce that the equations of the Gauss-Seidel system are:

$$\begin{aligned} x_1^{(k)} &= -0.75x_2^{(k-1)} + 6 \\ x_2^{(k)} &= -0.75x_1^{(k)} + 0.25x_3^{(k-1)} + 7.5 \\ x_3^{(k)} &= 0.25x_2^{(k)} - 6 \end{aligned}$$

Example 2 SOR with $\omega = 1.25$

- After computing the matrices

$$(D-\omega L)^{-1}, \qquad (1-\omega)D+\omega U$$

$$\begin{pmatrix} \frac{1}{4} & 0 & 0 \\ -\frac{15}{64} & \frac{1}{4} & 0 \\ -\frac{75}{1024} & \frac{5}{64} & \frac{1}{4} \end{pmatrix} \qquad \begin{pmatrix} -1 & -\frac{15}{4} & 0 \\ 0 & -1 & \frac{5}{4} \\ 0 & 0 & -1 \end{pmatrix}$$

- And computing the vector $c_s = \omega(D-\omega L)^{-1}b$

$$\left(-6, \frac{105}{8}, -\frac{243}{128}\right)^T$$

We find the SOR system $x^{(k)} = T_s x^{(k-1)} + c_s$

$$x_1^{(k)} = -0.25x_1^{(k-1)} - 0.9375x_2^{(k-1)} + 7.5$$
$$x_2^{(k)} = -0.9375x_1^{(k)} - 0.25x_2^{(k-1)} + 0.3125x_3^{(k-1)} + 9.375$$
$$x_3^{(k)} = 0.3125x_2^{(k)} - 0.25x_3^{(k-1)} - 7.5$$

Convergence comparison

Gauss-Seidel Iterations

k	0	1	2	3	$\cdots$	7
$x_1^{(k)}$	1	5.250000	3.1406250	3.0878906		3.0134110
$x_2^{(k)}$	1	3.812500	3.8828125	3.9267578		3.9888241
$x_3^{(k)}$	1	−5.046875	−5.0292969	−5.0183105		−5.0027940

SOR Iterations ($\omega = 1.25$)

k	0	1	2	3	$\cdots$	7
$x_1^{(k)}$	1	6.312500	2.6223145	3.1333027		3.0000498
$x_2^{(k)}$	1	3.5195313	3.9585266	4.0102646		4.0002586
$x_3^{(k)}$	1	−6.6501465	−4.6004238	−5.0966863		−5.0003486

General results

- **Theorem (Ostrowski-Reich)**
  If A is a positive definite matrix (正定値行列） then $\rho(T_\omega) < 1$ (convergence 収束する) for any value of $0 < \omega < 2$

- Better result is known if A is moreover tridiagonal (その上、三重対角行列だったらよりよい結果が知られている)

- In general, it is difficult to choose a good ω.

Subsection 2: Conjugate Gradient method

- Works only for positive definite matrix (正定値行列だけ)
- Work best for sparse matrix like tridiagonal
  三重対角行列のような疎行列に適切である。

- We will describe the basic method but to be efficient, more care is necessary.基礎な解法を記述するが、実用化するために、よりよく手配が必要：
- Because problems occur when A has small and large eigenvalues (that is has a large condition number)
  なぜならば、Aは大きいとともに小さな固有値があるときに問題が生じ得る（条件数が大きいという）
- This problem cannot be known in advance. Some "preconditioning" is necessary。
  予め、条件数の大きさを測定するのは難しい、大分ほとんどの場合、前処理を行う。