数理逍遥 II[16C1115] Topics in Mathematics。
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# 連立 1 次方程式系、反復法：練習問題

**Exercise 1** Consider the two following linear systems $(S_A)$ and $(S_B)$:

$$(S_A)\begin{cases} 4x_1 + x_2 - x_3 &= 5 \\ -x_1 + 3x_2 + x_3 &= -4 \\ 2x_1 + 2x_2 + 5x_3 &= 1 \end{cases} \qquad (S_B)\begin{cases} -2x_1 + x_2 + \frac{1}{2}x_3 &= 4 \\ x_1 - 2x_2 - \frac{1}{2}x_3 &= -4 \\ x_2 + 2x_3 &= 0 \end{cases}$$

1. Write the general iteration equation for (1) the Jacobi method and (2) for the Gauss-Seidel method.
2. Compute the first iteration $\mathbf{x}^{(1)} = (x_1^{(1)}, x_2^{(1)}, x_3^{(1)})^T$ and the second one $\mathbf{x}^{(2)} = (x_1^{(2)}, x_2^{(2)}, x_3^{(2)})^T$.

   Use the initial vector $\mathbf{x}^{(0)} = (0, 0, 0)^T$.
3. Write the system $(S_A)$ in matrix form $A \cdot x = b$. Write the decomposition $A = D - L - U$ in diagonal/lower triangular/upper triangular matrix.
4. Write down the matrix form of the Jacobi iteration: $\mathbf{x}^{(k+1)} = T_j\mathbf{x}^{(k)} + c_j$ and of the Gauss-Seidel $\mathbf{x}^{(k+1)} = T_g\mathbf{x}^{(k)} + c_g$.
5. Compute the characteristic polynomial $\chi_{T_j}(\lambda)$ of the matrix $T_j$ and $\chi_{T_g}(\lambda)$ the one of the matrix $T_g$.

   Deduce the spectral radii $\rho(T_j)$ and $\rho(T_g)$ of the matrices $T_j$ and $T_g$.
6. From Chapter 3, Section 2, Theorem 3, tell whether the Jacobi method converges. Same question for the Gauss-Seidel method.

**Exercise 2** *(Review on convergence results)*

1. Tell a necessary and sufficient condition for the Jacobi's and Gauss-Seidel's iteration method to converge.
2. In general, is this condition easy to verify in practice ?
3. What are the two cases (one is in report 3-1 !) we have seen for which it is "easy" to check the convergence ?
4. Could we tell *easily* that the Jacobi and Gauss-Seidel would converge for the systems $(S_A)$ and $(S_B)$ ?

**Exercise 3** Let $A \in \mathrm{Mat}_{n\times n}(\mathbb{R})$ , $\lambda \in \mathbb{C}$ an eigenvalue and $x \in \mathbb{C}^n$ one of its eigenvector. Show the following:

1. If $x$ is an eigenvectorof eigenvalue $\lambda$ of the matrix $A$, then $\lambda^k$ is an eigenvalue of the matrix $A^k$ and $x$ is an eigenvector for $\lambda^k$.
2. If $A$ is invertible, then $\lambda \neq 0$ and $\frac{1}{\lambda}$ is an eigenvalue of $A^{-1}$ with eigenvector $x$.
3. If all eigenvalues of $A$ verify $|\lambda| < 1$ (that is $\rho(A) < 1$) then $A - \mathrm{Id}$ is invertible and its eigenvalues are $\frac{1}{\lambda-1}$.

**Exercise 4** Assume that $A \in \mathrm{Mat}_{n\times n}(\mathbb{R}) \subset \mathrm{Mat}_{n\times n}(\mathbb{C})$ is symmetric, $\lambda \in \mathbb{C}$ an eigenvalue with an associated eigenvector $x \in \mathbb{C}^n$.

The purpose of this exercice is to show that $\lambda \in \mathbb{R}$ (and as a consequence that $x \in \mathbb{R}^n$)

1. Show that $\overline{y}^T A y = y^T A \overline{y}$ for any $y \in \mathbb{C}^n$
2. Write $\overline{x}^T A x - x^T A \overline{x}$ in function of $\lambda, \overline{\lambda}, x, \overline{x}$. Deduce that $\lambda = \overline{\lambda}$.

**Exercise 5** Gram-Schmidt orthogonalization.

Let $\langle\, .\,,\, .\,\rangle$ be an inner product on $\mathbb{R}^n$. Two vectors are orthogonal with respect to $\langle\, .\,,\, .\,\rangle$ if $\langle v_1, v_2\rangle = 0$.

Example: The dot product: $\langle a, b\rangle = a^T b$ for $a, b \in \mathbb{R}^n$.

1. Given two linearly independent vectors $v_1$, $v_2$ in $\mathbb{R}^2$, show that

$$e_1 = v_1 \tag{1}$$
$$e_2 = v_2 - \frac{\langle v_2, e_1\rangle}{\langle e_1, e_1\rangle} e_1 \tag{2}$$

   are two linealry independent orthogonal vectors.

2. More generally, given any linearly independent vectors $v_1, \ldots, v_n$ in $\mathbb{R}^n$, the *Gram-Schmidt* orthogonalization refers to the following computation of vectors $e_1, \ldots, e_n$.

$$e_1 = v_1 \tag{3}$$
$$e_k = v_k - \sum_{\ell=1}^{k-1} \frac{\langle v_k, e_\ell\rangle}{\langle e_\ell, e_\ell\rangle} e_\ell, \qquad \text{for } k = 2, \ldots n. \tag{4}$$

   Check that $\langle e_i, e_j\rangle = 0$ for $i \neq j$ (the family $e_1, \ldots, e_n$ is orthogonal).

3. Example: Consider $v_1 = (1, 0, 0)^T$, $v_2 = (1, 1, 0)^T$, $v_3 = (1, 1, 1)^T$. Orthogonalize the family $(v_1, v_2, v_3)$.

**Exercise 6** Orthogonal matrix

**Definition 1** *A matrix $Q \in \mathrm{Mat}_{n\times n}(\mathbb{R})$ is* orthogonal *if its columns are pairwise orthogonal (for the natural dot product) and of norm 1.*

1. Show that the definition means: $Q^T Q = \mathrm{Id}$ (thus an equivalent definition is $Q^T = Q^{-1}$).

2. Show that if $Q$ is an orthogonal matrix then, $\|Qx\|_2 = \|x\|_2$. Deduce that $|||Q|||_2 = 1$.

**Exercise 7** The purpose is to show the *spectral theorem*:

**Theorem 1** *Let $A \in \mathrm{Mat}_{n\times n}(\mathbb{R})$ be a symmetric matrix. There exist an orthogonal basis of eigenvectors of $A$.*

1. According to Exercise 4, $A$ has only real eigenvalues. Assume first that $A$ has at least two eigenvalues $\lambda_i \neq \lambda_j$, and let $x_i$, (respectively $x_j$) be an eigenvector for $\lambda_i$ (respectively for $\lambda_j$).

   Show that $x_i$ and $x_j$ are orthogonal, that is $x_i^T x_j = 0$.

2. Deduce that if $A$ has no multiple eigenvalue then Theorem 1 is true.

<u>General case (more difficult)</u>: Proof by induction on $n$.

3. How is the case $n = 1$ ?

   - Assume now the theorem true for any symmetric matrix of size $\leq n-1$.
   - Let $\lambda \in \mathbb{R}$ be an eigenvalue and $x \in \mathbb{R}^n$ be an associated eigenvector of norm 1, $\|x\|_2 = 1$.
   - Let $\langle x\rangle^T$ be the orthogonal complement of the line $\langle x\rangle$: $\langle x\rangle \oplus \langle x\rangle^\perp = \mathbb{R}^n$.
   - Let $y_2, \ldots, y_n$ be an orthonormal basis of $\langle x\rangle^\perp$ (obtained by Gram-Schmidt in Exercise 5 for example), so that $x, y_2, \ldots, y_n$ is an orthonormal basis of $\mathbb{R}^n$.

4. Let $y \in \langle x\rangle^\perp$. Show that $Ay \in \langle x\rangle^\perp$.

Let $P$ be the change of basis matrix, from the canonical basis of $\mathbb{R}^n$ to the basis $x, y_2, \ldots, y_n$:

$$P. \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = x, \qquad P. \begin{pmatrix} 0 \\ \vdots \\ 1 \text{ j-th entry} \\ \vdots \\ 0 \end{pmatrix} = y_j$$

5. Show that the matrix $P$ is orthogonal.

6. Show that $P^{-1}AP = B'$ where

$$B' = \begin{pmatrix} \lambda & \cdots & 0 & \cdots \\ \vdots & & & \\ 0 & & B & \\ \vdots & & & \end{pmatrix}$$

   where $B \in \mathrm{Mat}_{n-1\times n-1}(\mathbb{R})$ is a *symmetric* matrix.

7. Use induction hypothesis on $B$ to prove there is an *orthogonal* family of *eigenvectors* $(1, 0, \ldots, 0)^T, q_2, \ldots, q_n \in \mathbb{R}^n$ for $B'$, which is a *basis* of $\mathbb{R}^n$.

8. Conclude the proof of Theorem 1

**Exercise 8** Prove the following simple Corollary of Theorem 1.

**Corollary 1** *A matrix $A \in \mathrm{Mat}_{n\times n}(\mathbb{R})$ is symmetric, if and only if there exist an orthogonal matrix $Q$ and a diagonal matrix $D$ such that $A = Q^T DQ$.*