Essential Mathematics for Global Leaders I

Lecture 6-2

Differential Equations II

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Plan (tentative)

[4/13] L1 : introduction. Review of high-school mathematics in English.

[4/20-27] L2-3 : Functions and graphs. Plotting with Mathematica I (グラフをプロットする)

[5/7] L4 : Infinitely small and large : limits (極限)

[5/11] L5 : Differentiation (連続性と微分法)

[5/18] L6 : Differentiation II : extrema, related rates ...（極値と...）

[5/25] L7 : Differentiation III : Newton’s method, Taylor’s expansion (ニュートン法とテイラー展開）

[6/1] L8 : Mid-term test. Integration I : definition, fundamental theorem of calculus　積分I.

[6/8] L9 : computation of indefinite integrals　不定積分

[6/15] L10 : Application of Integration I : length, volume and surfaces
積分の応用：長さ、面積、体積

[6/22] L11 : Application of Integration II : average, center of mass （質量中心）~~, work of a force.~~

[6/29] L12 : Ordinary Differential equations (one variable)　常微分方程式

[7/6] L13 : Linear Differential Equations of order 2 : harmonic oscillators (small-angle pendulum, spring). 二階線形微分方程式：調和振動子（振幅が小さい振り子、ばね)

[7/13] L14 : Ordinary Differential Equations with Mathematica. Mathematicaを利用して常微分

ODE II: content

1. 2^{nd} order linear differential equation
２階線形常微分方程式

2. 2^{nd} order linear equation and harmonic oscillator
２階線形常微分方程式と調和振動子

3. Solving 2^{nd} order linear homogeneous equations
斉次な2階線型常微分方程式を解く

4. Interpretation of solutions to harmonic oscillators with damping.　その解が減衰調和振動子に与える解釈

5. Solving non-homogenous 2^{nd} order linear equation.
Harmonic oscillator with forced vibration.
斉次でない2階線形常微分方程式を解く。強制調和振動子。

2^{nd} order linear differential equation 二階線形微分方程式

- A solution $\phi(t)$ to the 2^{nd} order differential equation $y'' = f(t, y, y')$ satisfies:

$$\phi''(t) = f\big(t, \phi(t), \phi'(t)\big)$$

- Linear: $p(t)y'' + q(t)y + r(t)y = g(t)$
- Homogeneous（斉次な線形常微分方程式）

$$p(t)y'' + q(t)y + r(t)y = 0$$

- Constant coefficients: $ay'' + by' + cy = d$

- 注: there is no direction field ! (方向場がない）

2^{nd} order → 1st order system of 2 equations in 2 unknowns

- $y'' = f(t, y, y')$
- $y_1: E_1 \subset \mathbb{R} \to \mathbb{R}, \quad y_2: E_2 \subset \mathbb{R} \to \mathbb{R}$

$$Y = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}: E_1 \times E_2 \subset \mathbb{R}^2 \to \mathbb{R}^2$$

- Let $y_1(t) = y(t)$ and $y_2(t) = y'(t)$

$$Y' = \begin{pmatrix} y_1'(t) \\ y_2'(t) \end{pmatrix} = \begin{pmatrix} y'(t) \\ y''(t) \end{pmatrix} = F\big(t, Y(t)\big) = \begin{pmatrix} y_2(t) \\ f\big(t, y_1(t), y_2(t)\big) \end{pmatrix}$$

- If $f(t, y, y')$ is linear: $y'' = ay' + by + c$ then

$$Y' = \begin{pmatrix} 0 & 1 \\ b & a \end{pmatrix} Y + \begin{pmatrix} 0 \\ c \end{pmatrix}$$

is a linear system of differential equations.

Vector fields in "space-phase" (位相空間)

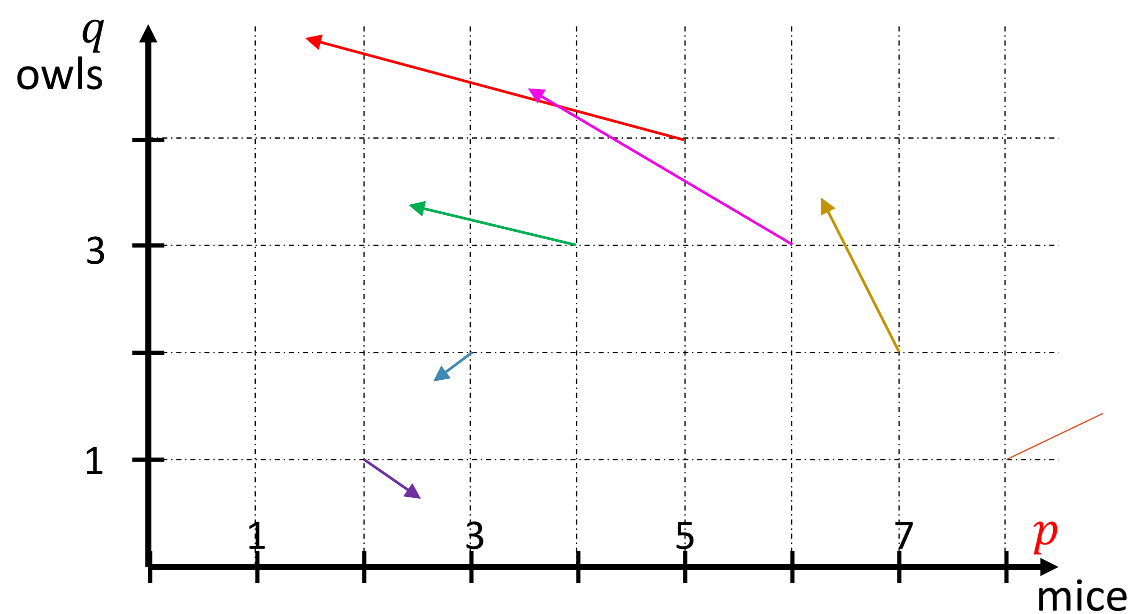
- Predator/prey equation (L6-1, Slide 22)

$$p'(t) = 0.5p(t) - 0.3q(t)p(t)$$
$$q'(t) = -0.7q(t) + 0.2q(t)p(t)$$

System of 2 differential equations in 2 unknowns p and q

- Let's find the vector field in "Space-phase"

p	q	p'	q'
2	1	0.4	-0.3
3	2	-0.3	-0.2
4	3	-1.6	0.3
5	4	-3.5	1.2
6	3	-2.4	1.4
7	2	-0.7	1.4
8	1	1.6	0.9



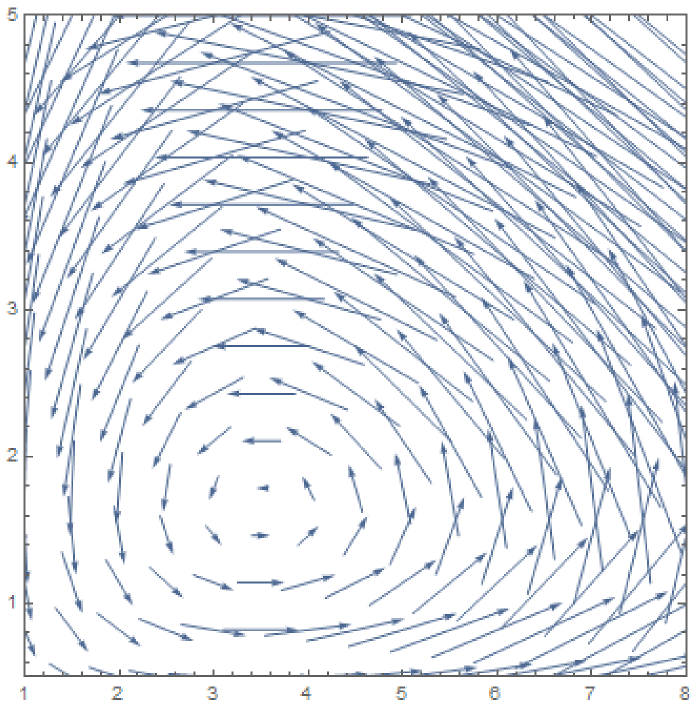
- Autonomous equation. If not, the vector field is animated (change with time)

Plotting the Space-Phase with Mathematica

```
VectorPlot[{0.5p-0.3p*q,-0.7q + 0.2 p*q } , {p,1,8} ,
{q,0.5,5} , VectorStyle -> Arrowheads[0.02] , PlotRange
-> {{1,8},{0.5,5}} , VectorScale -> 1]
```

```
VectorPlot[{0.5p-0.3p*q , -0.7 q + 0.2p*q} , {p,1,8} ,
{q,0.5,5} , VectorStyle->Arrowheads[0.02]]
```

owls



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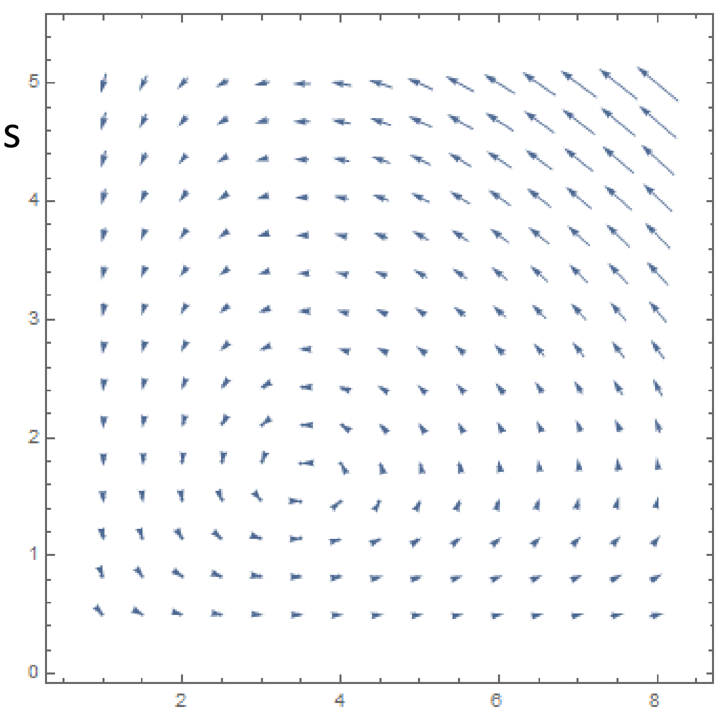

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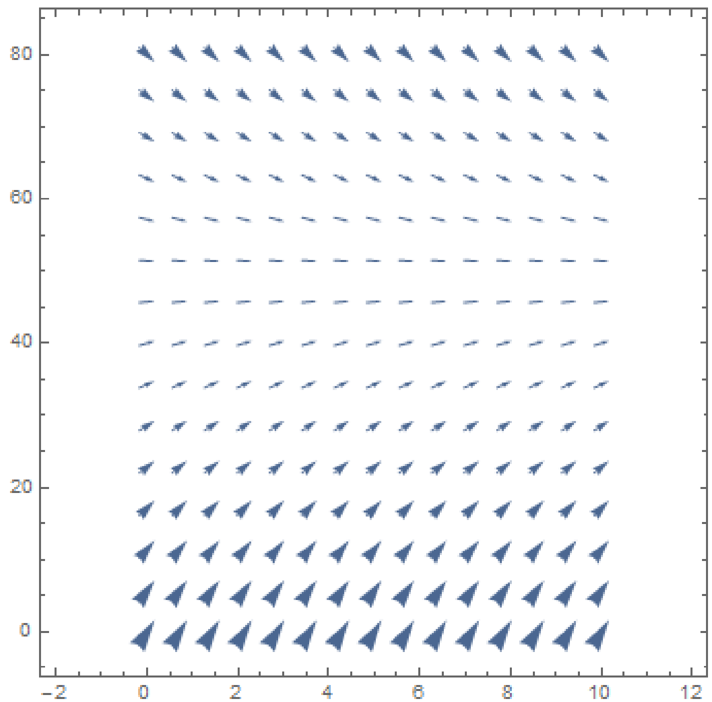
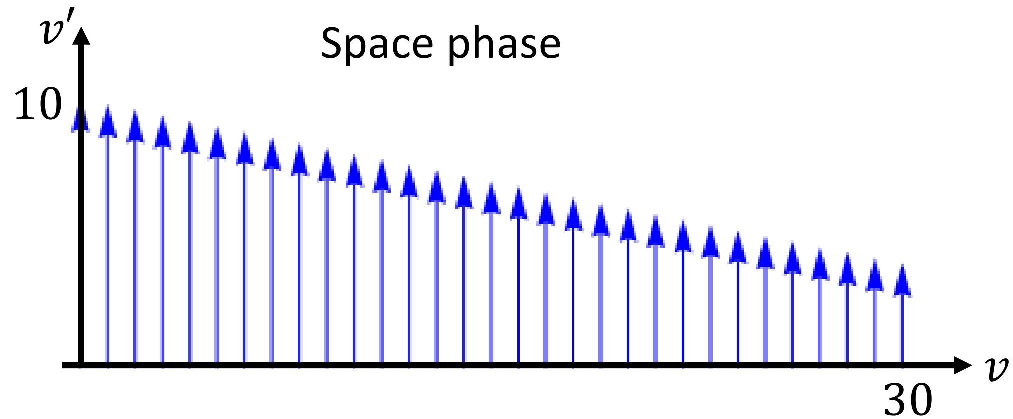
Direction field and vector field in the space-phase（方向場と位相空間でのベクトル場）

- 1^{st} order differential equation with 1 unknown
  $v' = 9.8 - 0.2v$ (L6-1, Slide 6)

Direction field (L6-1 slide 9)



Space phase



1 dimensional space-phase. It is never used in practice. “Space-phase” is use for systems of diff. eqn.
1次元の位相空間。具体的に使われていない。常微分方程式系の場合使われている。

- 2^{nd} order linear equation → use space-phase

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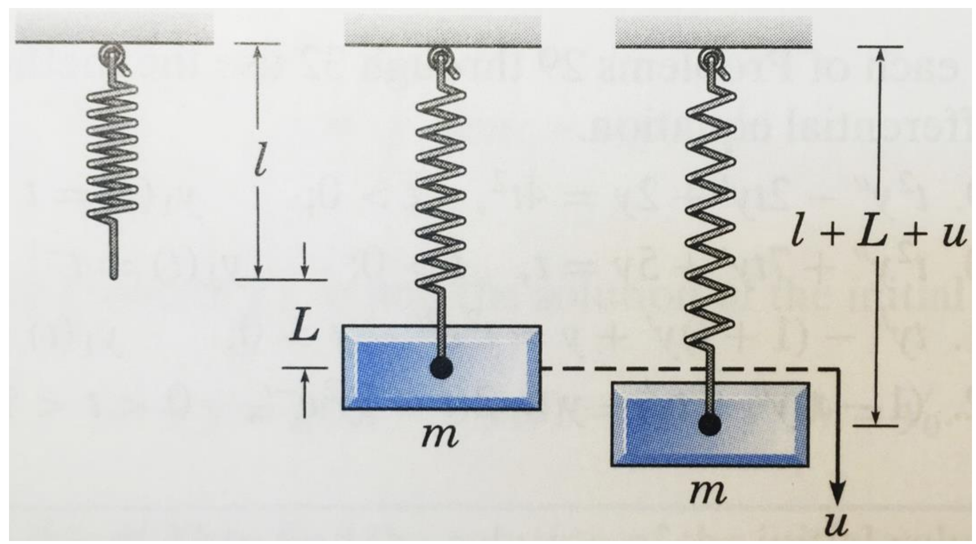
Typical example: Harmonic oscillators
代表例：調和振動子

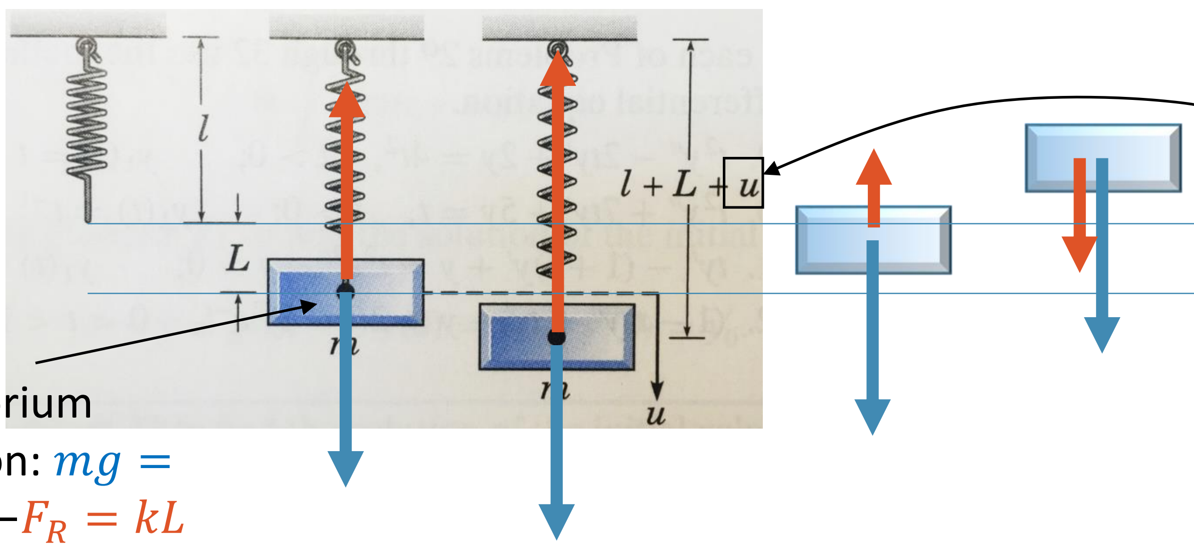
- **Definition**: System that when displaced from its equilibrium position, creates a restoring force
  定義：均衡位置から離すと、復元力が表れる。
- 例：Spring-mass system （ばね質量系):
- With/without damping (減衰力の有無）
- With/without external force (外力の有無）



- Another example ? RLC electric circuits (電気回路）

Spring-mass system : ばね質量系

- Newton's 2nd law: $m\,\vec{a} = \sum \overrightarrow{force}$
- Gravitational force F_G, restoring force F_R



$u(t)$:Displacement from the equilibrium position: 均衡位置から変位

Equilibrium position: $mg = F_G = -F_R = kL$

- **Hookes'law**: F_R is proportional to the $L + u(t)$

$$m\,u''(t) = mg - k\big(L + u(t)\big) = -ku(t)$$

Gravitational acceleration 重力加速度

k: string constant （ばね定数）

L: Distance between spring's natural length and equilibrium position 均衡位置とばねの自然な長さから距離

Simple harmonic oscillator自由振動子

Without damping nor external force (外力も減衰も無）

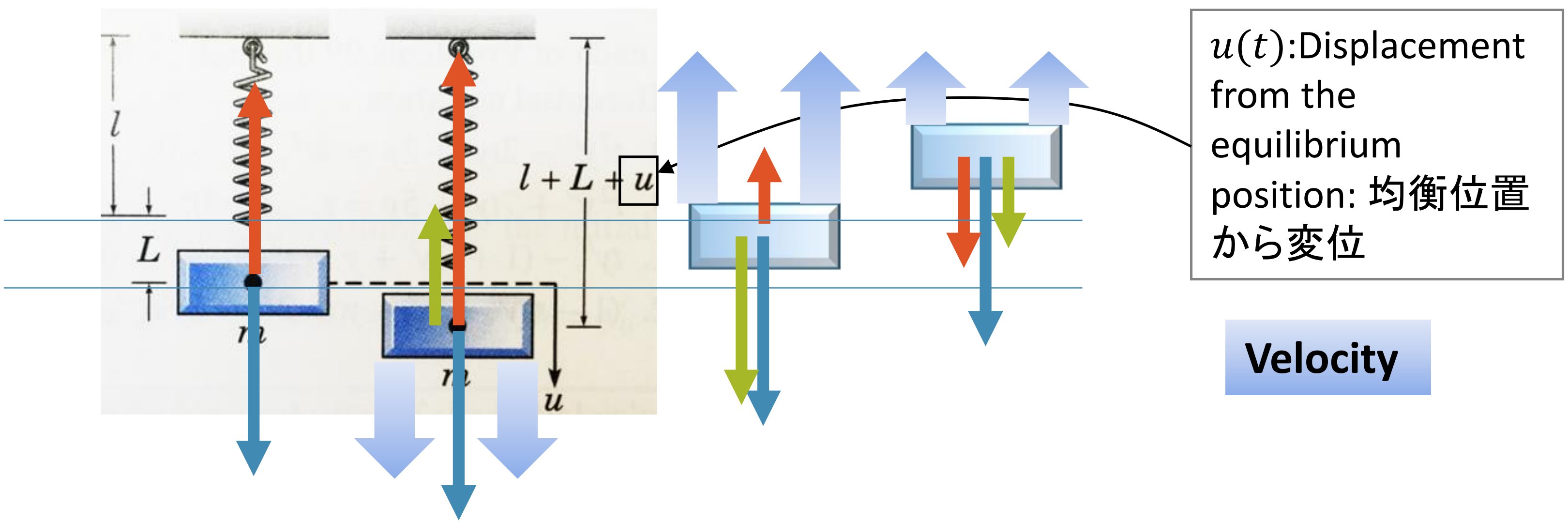
$$m\,u''(t) + ku(t) = 0$$

This is called “Simple harmonic oscillator”.

- Exercise: can you find solutions ?

Harmonic oscillator with damping
減衰調和振動子

- A damping force is opposed to the direction of motion. Usually the force is proportional to the speed. Here $F_D = -\gamma u'(x)$
(減衰力は運動に抵抗し、速度に比例することである)



Newton's law: $mu''(x) = F_G + F_R + F_D = -ku(t) - \gamma u'(x)$

$$mu''(t) + \gamma u'(t) + ku(t) = 0$$

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Solving homogeneous linear differential equations: use linearity!

- General principle for homogeneous linear equations (斉次な線形微分方程式に関する一般な原理）

$$a_0(t)\frac{d^n y}{dt^n} + a_1(t)\frac{d^{n-1}y}{dt^{n-1}} + \cdots + a_n(t)y = 0$$

- **Linearity properties of solutions** (解の線形性）
If $y_1(t)$ and $y_2(t)$ are two solutions then:
$y_1(t) + y_2(t)$ is also solution.
$\lambda y_1(t)$ is also solution, $\lambda \in \mathbb{R}$.

- Why? Because $\frac{d^n(y_1(t)+\lambda y_2(t))}{dt^n} = \frac{d^n y_1(t)}{dt^n} + \lambda\frac{d^n y_2(t)}{dt^n}$

Homogeneous 2nd order linear equation with constant coefficients
(定数系斉次な二階線形方程式)

$$\boxed{ay'' + by' + cy = 0}, \text{ with } a, b, c \in \mathbb{R}.$$

- **Fact**: The set of solutions is generated by two (linearly independent) functions.
  解の全体が(一次独立な)解の二つに生成される。

- Characteristic equation: $ax^2 + bx + c = 0$
  $$\Delta = b^2 - 4ac$$
- If $\Delta > 0$, solutions: $x_1, x_2 = \left(-b \pm \sqrt{\Delta}\right)/2a \in \mathbb{R}$

- **Theorem**: If $\Delta > 0$, the solutions of the diff. eq.
  $$ay'' + by' + cy = 0$$
  are of the form: $\boxed{Ae^{x_1 t} + Be^{x_2 t}}$ for any $A, B \in \mathbb{R}$.

Initial value and Uniqueness
初期値と一意性

- For 1st order linear differential equation, the condition $y(0) = y_0$ guarantees uniqueness.
一階線形微分方程式の場合、$y(0) = y_0$初期条件は一意性を与える。
- For 2nd order equations, we need two conditions:
$$y(0) = y_0, \qquad y'(0) = y_0'$$
- Exercise: Find the (unique) solution of:
$$y'' + 5y' + 6y = 0, \qquad y(0) = 2, \qquad y'(0) = 3$$

Characteristic equations: $x^2 + 5x + 6 = 0,\ \Delta = 25 - 24 = 1.$
$x_1, x_2 = \frac{-5 \pm 1}{2} = -3, -2 \Rightarrow$ General solutions: $y(t) = Ae^{-3t} + Be^{-2t}$
Initial Conditions: $y(0) = 2 \Rightarrow A + B = 2,\ \ y'(0) = 3 \Rightarrow -3A - 2B = 3$
$\mathrm{B} = 9, \qquad \mathrm{A} = -7 \qquad \boxed{y(t) = -7e^{-3t} + 9e^{-2t}}$

Case of complex solutions (II)

- $\Delta = b^2 - 4ac < 0, \qquad \Delta = i^2 \cdot (-\Delta)$
- Roots: $x_1, x_2 = \left(-b \pm i\sqrt{-\Delta}\right)/2a \in \mathbb{C}$
- Solutions: $y_1(t) = e^{x_1 t} \qquad y_2(t) = e^{x_2 t}$ …. Complex valued functions（複素値関数）
- Can we have real valued functions ? (実数値関数）
- $y_1(t) + y_2(t) = 2e^{-\frac{bt}{2a}} \cos\left(\frac{\sqrt{-\Delta}}{2a} t\right)$ is a real valued function（実数値関数）
- $y_1(t) - y_2(t) = e^{-\frac{bt}{2a}} \left(\cos\left(\frac{\sqrt{-\Delta}}{2a} t\right) + i\sin\left(\frac{\sqrt{-\Delta}}{2a} t\right) - \cos\left(-\frac{\sqrt{-\Delta}}{2a} t\right) - i\sin\left(-\frac{\sqrt{-\Delta}}{2a} t\right)\right) = 2\, i\, e^{-\frac{bt}{2a}} \sin\left(\frac{\sqrt{-\Delta}}{2a} t\right)$

Case of complex solutions (III)

- $u(t) = \big(y_1(t) - y_2(t)\big)/2i = e^{-\frac{bt}{2a}} \sin\left(\frac{\sqrt{-\Delta}}{2a} t\right) \in \mathbb{R}$
- $v(t) = \big(y_1(t) + y_2(t)\big)/2 = e^{-\frac{bt}{2a}} \cos\left(\frac{\sqrt{-\Delta}}{2a} t\right) \in \mathbb{R}$

- **Theorem:** When $\Delta < 0$, the solutions of $ay'' + by' + cy = 0$ are of the form
  $Au(t) + B\, v(t),$ for any A and B in $\mathbb{R}$

- **Exercise**: find the set of real valued functions, solutions of $y'' + y' + 9.25y = 0.$
  What is the solutions with initial values:
  $$y(0) = 2, y'(0) = 8\,.$$

- $y(t) = Ae^{-\frac{t}{2}} \sin(3t) + Be^{-\frac{t}{2}} \cos(3t).$ $y(0) = B = 2.$ $y'(0) = 3A - 1$
- $A = 3,$ $y(t) = 3e^{-\frac{t}{2}} \sin(3t) + 2e^{-\frac{t}{2}} \cos(3t)$

Case of double solutions $\Delta = 0$ (I)

- Example: $y'' + 4y' + 4y = 0$
- The characteristic equation $x^2 + 4x + 4$ has a $\Delta = 0$, and indeed is equal to $(x+2)^2$.
- The two solutions $y_1(t) = e^{-2t} = y_2(t)$ are equal and they don’t generate all solutions….
(二つ解は一致してしまい、すべての解を生成することはない)
- Idea:  Check that $te^{-2t} = ty_1(t)$ is also solution.

Case of double solutions $\Delta = 0$ (II)

- In general, if the characteristic equation of $ay'' + by' + cy = 0$ has a $\Delta = 0$, then the same trick works: （一般に同じ工夫が適用する）

- There is the solution: $y_1(t) = e^{-\frac{bt}{2a}}$
  We verify that $y_2(t) = ty_1(t)$ is also solution.

- **Theorem:** The set of solutions of the diff. eqn. $ay'' + by' + cy = 0$ when $b^2 - 4ac = 0$ is:

$$Ae^{-\frac{bt}{2a}} + B\,t\,e^{-\frac{bt}{2a}}, \qquad for\ any\ A, B \in \mathbb{R}$$

Exercise: Find the solution of :$y'' - y' + 0.25y = 0$, with initial conditions $y(0) = 2$, $y'(0) = 1/3$.

$y(t) = Ae^{\frac{t}{2}} + Bte^{\frac{t}{2}}$. $y(0) = A = 2$, $\quad y'(0) = 1 + B = \frac{2}{3}$

$$y(t) = 2e^{\frac{t}{2}} - \frac{2}{3}e^{\frac{t}{2}}$$

Review on the solution of $ay'' + by' + cy = 0\,,\ a,b,c \in \mathbb{R}$

- Characteristic equation: $ax^2 + bx + c = 0$
- If $\Delta > 0$, let $x_1 \neq x_2$ be the two roots.
  Solutions are $\boxed{Ae^{x_1 t} + Be^{x_2 t}}$ $\quad A, B \in \mathbb{R}$
- If $\Delta < 0$, let $x_1 = \lambda + i\mu$, $x_2 = \lambda - i\mu$ be the two distinct complex roots.
  The real valued solutions are:
  $$\boxed{Ae^{\lambda t}\cos(\mu t) + Be^{\lambda t}\sin(\mu t)} \quad A, B \in \mathbb{R}$$
- If $\Delta = 0$, then $x_1 = -b/2a$ and the solutions are:
  $$\boxed{Ae^{x_1 t} + Bte^{x_1 t}} \quad A, B \in \mathbb{R}$$

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Back to simple harmonic oscillator (no damping page 11-12)

- Free mass-spring system: $mu''(t) + ku(t) = 0$

$$x^2 + k/m = 0, \qquad x_1, x_2 = \pm i\sqrt{k/m} \coloneqq \pm i\omega_0$$

$$u(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t)$$

Periodic motion (周期運動)

$\omega_0 \rightarrow$ Natural frequency (周波数)

$T = 2\pi/\omega_0 \rightarrow$ period (周期)

$R = \sqrt{A^2 + B^2} \rightarrow$ amplitude (振幅)

- No damping $\rightarrow$ perpetual motion（永久運動）

Harmonic oscillator with damping

- Damped spring-mass system:

$$mu''(t) + \gamma u'(t) + ku(t) = 0$$

k: string constant （ばね定数）

γ: damping coefficient ダンパの減衰係数

Characteristic equation $\Delta = \gamma^2 - 4mk$

$$r_1, r_2 = \left(-\gamma \pm \sqrt{\Delta}\right)/2m$$

- Depending on the sign of $\gamma^2 - 4mk = \Delta$

$\Delta > 0$ $\quad u = Ae^{r_1 t} + Be^{r_2 t}$

$\Delta = 0$ $\quad u = (A + Bt)e^{-\gamma t/2m}$

$\Delta < 0$ $\quad \begin{cases} u = e^{-\gamma t/2m}(A\cos(\omega_d t) + B\sin(\omega_d t)) \\ \text{where} \quad \omega_d = \sqrt{-\Delta}/2m > 0 \end{cases}$

$\Delta < 0$: Under Damped oscillator

Mass （質量）$m = 1$

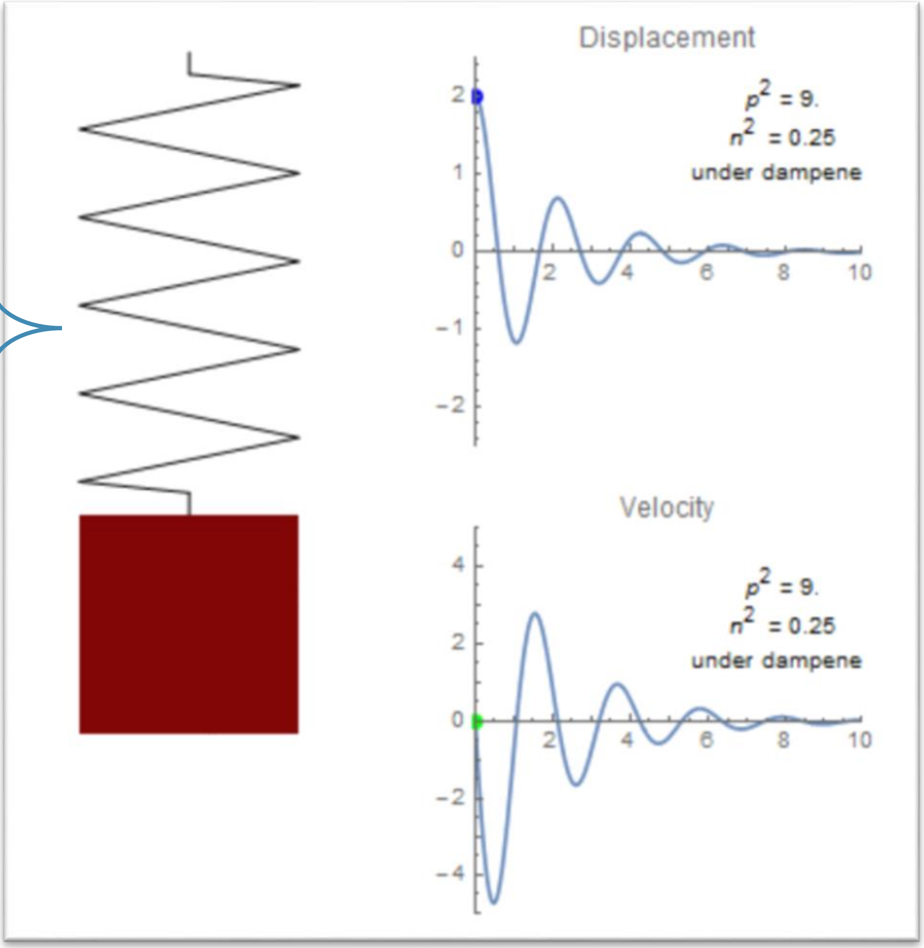
Damping coefficient: $\gamma = 1$

（ダンピンの減衰係数）

Spring's constant: $k = 9$

（ばねの定数）

$$\gamma^2 - 4km = 1 - 36 < 0$$



The system still oscillates at the "pseudo-frequency" $\omega_d = \sqrt{-\Delta}/2m$

系はまだ振動している

$$\omega_d = \omega_o \sqrt{1 - \frac{\gamma^2}{4km}}$$ 減衰固有角振動数（ω_0固有角振動数）

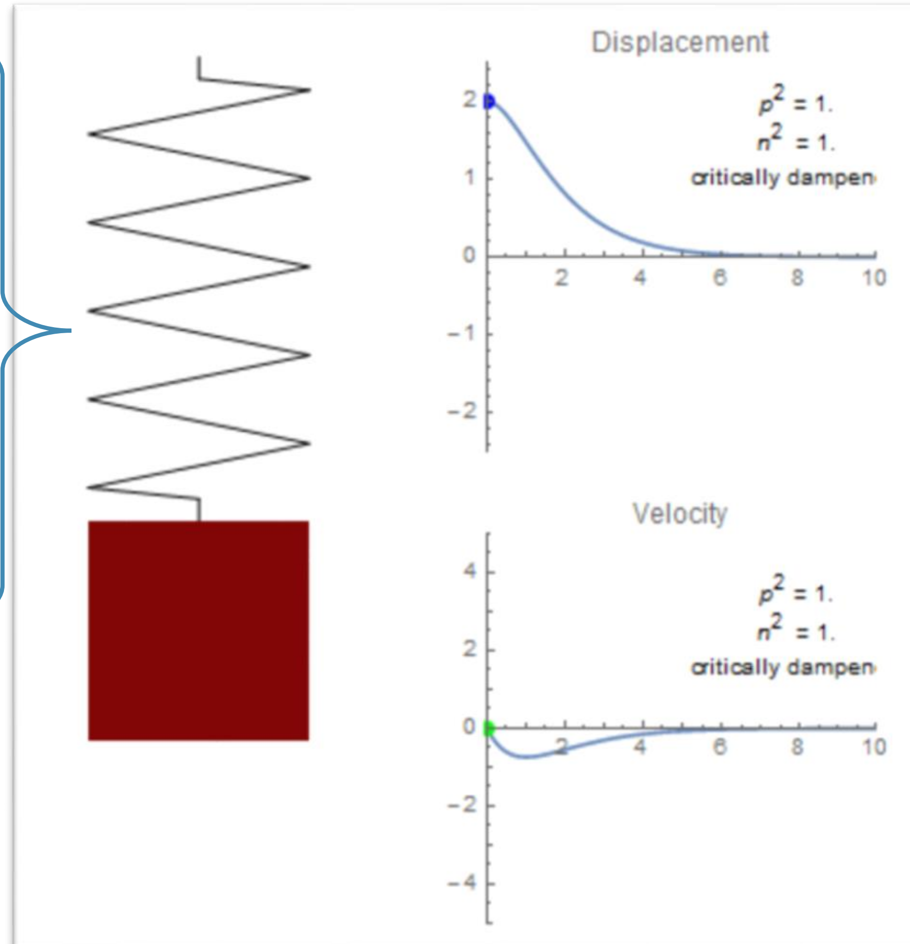
$\Delta = 0$ Critically damped oscillator 臨界減衰

Mass （質量）$m = 1$

Damping coefficient: $\gamma = 2$
（ダンピンの減衰係数）

Spring’s constant: $k = 1$
（ばねの定数）

$$\gamma^2 - 4km = 4 - 4 = 0$$



The system doesn’t oscillate anymore. 系はもう振動しない。

The mass is going to its equilibrium state the fastest.
量が平衡状態に戻る速度は最も速い。

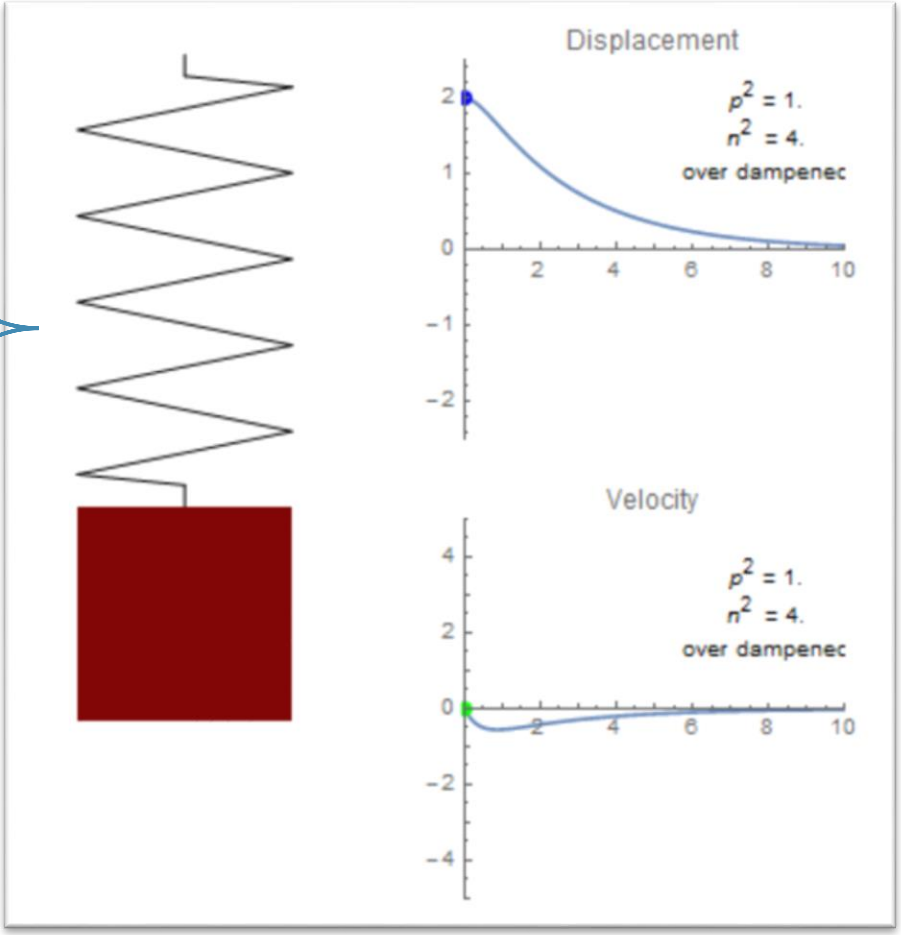
$\Delta > 0$ Over-damped oscillator　過減衰

Mass （質量）$m = 1$

Damping coefficient: $\gamma = 3$
（ダンピンの減衰係数）

Spring's constant: $k = 1$
（ばね定数）

$\gamma^2 - 4km = 9 - 4 = 5 > 0$

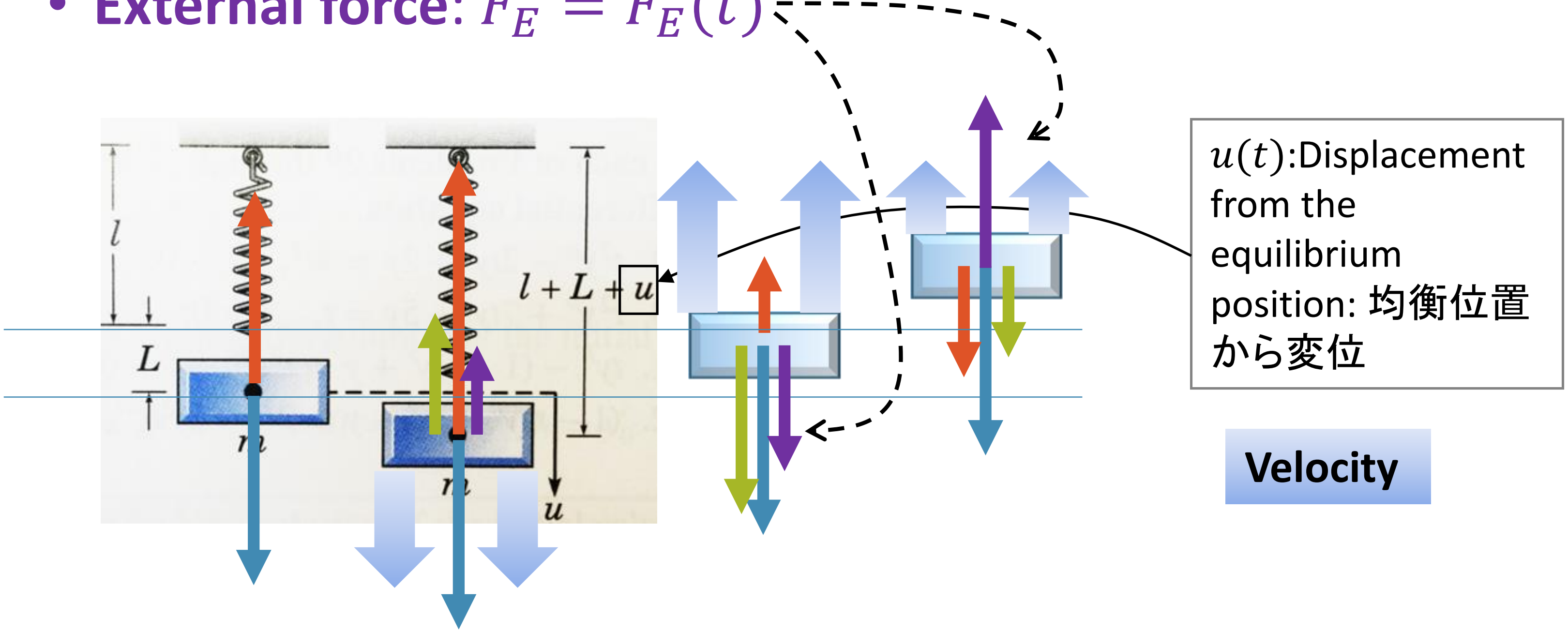


The system doesn't oscillate anymore.　系はもう振動しない。

The mass is going to its equilibrium state slower.
量が平衡状態により遅く戻る。

Forced oscillations 強制振動

- Gravitational force $F_G = mg$,
- Restoring force $F_R = -k\left(L + u(t)\right)$ $(kL = mg)$
- damping force: opposed to the direction of motion.
  $F_D = -\gamma u'(x)$
- **External force**: $F_E = F_E(t)$



Newton's law: $mu''(t) = F_G + F_R + F_D + F_E = -ku(t) - \gamma u'(x)$

$$mu''(t) + \gamma u'(t) + ku(t) = F_E(t)$$

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Non-homogeneous 2nd order linear equation 斉次でない2階線形常微分方程式

$$ay'' + by' + cy = F_E(t)$$

- For homogeneous equations, the set of solutions is generated by two $y_1(t)$ and $y_2(t)$ independent solutions: $Ay_1(t) + By_2(t), \quad A, B \in \mathbb{R}$
  斉次な方程式の場合、解の全体が二つの独立な解$y_1(t)$と $y_2(t)$ に生成される。

And non-homogeneous? If we found one particular solution,（特殊解）$Y(t)$ then the set of solutions is:

$$Ay_1(t) + By_2(t) + Y(t), \qquad A, B \in \mathbb{R}$$

一般解 (general solution) — $Ay_1(t) + By_2(t)$

特殊解 (particular solution) — $Y(t)$

Particular solution + general solution
特殊解 + 一般解

- Why? If $Y_1(t)$ and $Y_2(t)$ are particular solutions:

$$
\begin{aligned}
& aY_1''(t) + bY_1'(t) + cY_1(t) = F_E(t) \\
- \quad & aY_2''(t) + bY_2'(t) + cY_2(t) = F_E(t)
\end{aligned}
$$

$$a\left(Y_2''(t) - Y_1''(t)\right) + b\left(Y_2'(t) - Y_1'(t)\right) + c\left(Y_2(t) - Y_1(t)\right) = 0$$

⇒ So $Y_2(t) - Y_1(t)$ is solution of the homogeneous differential equation (斉次な常微分方程式の解）

$$ay'' + by' + cy = 0 \Rightarrow Y_2 - Y_1 = \alpha y_1(t) + \beta y_2(t)$$

- We know how to find general solutions (一般解）
  How to find a particular solution ? (特殊解）

Finding a particular solution: method of undetermined coefficients (case: e^{2t})
特殊解を求める:未定係数法　(case: e^{2t})

- Example: $y'' - 3y' - 4y = 3e^{2t}$
  A possible particular solution may be $Y(t) = Ae^{2t}$.
  Is it true ? Find $A \in \mathbb{R}$.

$$Y'(t) = 2Ae^{2t}, \qquad Y''(t) = 4Ae^{2t},$$
$$Y'' - 3Y' - 4Y = e^{2t}(4A - 6A - 4A) = -6Ae^{2t}$$

$\Rightarrow A = -1/2,$ and a particular solution is $Y(t) = -1/2\ e^{2t}$

- Find the set of all solutions.

Characteristic equation: $x^2 - 3x - 4 = 0.$ $\Delta = 9 + 16 = 25.$
$x_1, x_2 = 3 \pm 5/2 = 4\ or\ -1.$
The solutions of the homogeneous equation $y'' - 3y' - 4y = 0$ are $y(t) = ae^{4t} + be^{-t}$ $(a, b \in \mathbb{R})$ therefore the solutions are given by: $\boxed{ae^{4t} + be^{-t} - \frac{1}{2}e^{2t}}$

Method of undetermined coefficients
(case of sin or cos)　未定係数法

- $y'' - 3y' - 4y = \sin(t)$
  What is a possible particular solution?

Let's try: $Y(t) = a\sin(t) + b\cos(t)$

$Y'(t) = a\cos(t) - b\sin(t)$,　　$Y''(t) = -a\sin(t) - b\cos(t)$

Thus: $Y'' - 3Y' - 4Y = \sin(t)\,(-a + 3b - 4a) + \cos(t)\,(-b - 3a - 4b)$

And we find

$$\begin{cases} 3b - 5a = 1 \\ -3a - 5b = 0 \end{cases} \qquad \begin{cases} 3b + \dfrac{25}{3}b = 1 \\ a = -\dfrac{5}{3}b \end{cases} \qquad \begin{cases} b = \dfrac{3}{34} \\ a = -\dfrac{5}{3}b \end{cases}$$

$Y(t) = -\frac{5}{3}\sin(t) + \frac{3}{34}\cos(t)$

The set of all solution is: $\boxed{\left\{ y(t) = ae^{-t} + be^{2t} - \frac{5}{3}\sin(t) + \frac{3}{34}\cos(t) \mid a, b \in \mathbb{R} \right\}}$

Method of undetermined coefficients (case of polynomial) 未定係数法 (多項式のケース)

- $y'' - 3y' - 4y = 4t^2 - 1$
- A particular solution is a polynomial of degree 2:

$$Y(t) = a_2 t^2 + a_1 t + a_0$$

$$Y'(t) = 2a_2 t + a_1, \qquad Y''(t) = 2a_2$$

$$Y'' - 3Y' - 4Y = t^2(-4a_2) + t(-4a_1 - 6a_2) + 2a_2 - 3a_1 - 4a_0$$

$$\begin{cases} -4a_2 = 4 \\ -4a_1 - 6a_2 = 0 \\ 2a_2 - 3a_1 - 4a_0 = -1 \end{cases} \qquad \begin{cases} a_2 = -1 \\ -4a_1 + 6 = 0 \\ -2 - 3a_1 - 4a_0 = -1 \end{cases} \qquad \begin{cases} a_2 = -1 \\ a_1 = 3/2 \\ -9/2 - 4a_0 = 1 \end{cases}$$

$$\begin{cases} a_2 = -1 \\ a_1 = 3/2 \\ a_0 = -11/8 \end{cases}$$

$$Y(t) = -t^2 + \frac{3}{2}t - 11/2$$

All solutions: $y(t) = ae^{-t} + be^{4t} - t^2 + \frac{3}{2}t - \frac{11}{2}$

未定係数法: case of a product of $e^{\alpha t}$ with/or cos or with sin or with polyn.

- $y'' - 3y' - 4y = -8e^{t}\cos(2t)$

A possible particular solution is of the form

$$Y(t) = Ae^{t}\cos(2t) + Be^{t}\sin(2t)$$

$$Y'(t) = e^{t}(A\cos(2t) + B\sin(2t)) + e^{t}(-2A\sin(2t) + 2B\cos(2t))$$

$$Y''(t)$$
$$= e^{t}(A\cos(2t) + B\sin(2t)) + e^{t}(-2A\sin(2t) + 2B\cos(2t))$$
$$+ e^{t}(-4A\cos(2t) \ - 4\,B\sin(2t))$$

$$\begin{cases} e^{t}\cos(t)\,(A + 4B - 4A - 3A - 6B - 4A) = -8 \\ e^{t}\sin(t)\,(B - 4A - 4B - 3B + 6A - 4B) = 0 \end{cases} \qquad \begin{cases} -10A - 2B = -8 \\ -10\mathrm{B} + 2A = 0 \end{cases}$$

$$\begin{cases} -52B = -8 \\ -10\mathrm{B} + 2A = 0 \end{cases} \qquad \begin{cases} B = 2/13 \\ -20/13 + 2A = 0 \end{cases} \qquad \begin{cases} B = 2/13 \\ A = 10/13 \end{cases}$$

$$\boxed{Y(t) = \frac{10}{13}e^{t}\cos(2t) + \frac{2}{13}\,e^{t}\sin(2t)}$$

未定係数法：case of sum of functions cos, sin, $e^{\alpha t}$, polynomial

- $y'' - 3y' - 4y = -8e^{t}\cos(2t) + 3e^{2t}$

Slide 36 -> Particular solution of $y'' - 3y' - 4y = -8e^{t}\cos(2t)$ is

$$Y(t) = \frac{10}{13}e^{t}\cos(2t) + \frac{2}{13}\,e^{t}\sin(2t)$$

Slide 33 -> Particular solutions of $y'' - 3y' - 4 = -8e^{t}\cos(2t)$ is

$$Y(t) = -1/2\;\; e^{2t}$$

Slide 15 -> Particular solutions of $y'' - 3y' - 4y = -8e^{t}\cos(2t) + 3e^{2t}$ is

$$\boxed{Y(t) = \frac{10}{13}e^{t}\cos(2t) + \frac{2}{13}\,e^{t}\sin(2t) - \frac{1}{2}e^{2t}}$$

Slide 32 -> Any solution of $y'' - 3y' - 4y = -8e^{t}\cos(2t) + 3e^{2t}$ is

$$\boxed{\mathrm{y}(t) = \frac{10}{13}e^{t}\cos(2t) + \frac{2}{13}\,e^{t}\sin(2t) - \frac{1}{2}e^{2t} + ae^{-t} + b\,e^{4t}, \quad \text{for } a, b \in \mathbb{R}}$$

未定係数法： まとめ summary

- $ay'' + by' + cy = g_i(t)$
- A possible particular solution $Y_i(t)$ is of the form: (where undetermined coefficients A_j must be found)
- 注 : if $Y_i(t)$ is a general solution of the homogeneous equation $ay'' + by' + c = 0$ then the method fails. もしこうして定義された解$Y_i(t)$が斉次な方程式の解であったら、方法は失敗する。

	$g_i(t)$	$Y_i(t)$
Polynomial	$a_n t^n + a_{n-1} t^{n-1} + \cdots + a_o$	$A_n t^n + A_{n-1} t^{n-1} + \cdots + A_0$
Polynomial × exponential	$(a_n t^n + a_{n-1} t^{n-1} + \cdots$	$(A_n t^n + A_{n-1} t^{n-1} + \cdots$
Polynomial × exponential × sin or cos	$(a_n t^n + \cdots + a_o)e^{\alpha t} \begin{cases} \sin(\beta t) \\ \cos(\beta t) \end{cases}$	$(A_n t^n + \cdots \begin{cases} \sin(\beta t) \\ \cos(\beta t) \end{cases}$

Homework 練習

1. Find the solutions of the following diff. eq
   a. $y'' - 2y' - 3y = 3e^{2t}$
   b. $y'' + 2y' + 5y = 3\sin(2t)$
   c. $2y'' + 3y' + y = t^2 + 3sin(t)$