Essential Mathematics for Global Leaders I

Lecture 6-1

Differential Equations I

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Plan (tentative)

[4/13] L1 : introduction. Review of high-school mathematics in English.

[4/20-27] L2-3 : Functions and graphs. Plotting with Mathematica I (グラフをプロットする)

[5/7] L4 : Infinitely small and large : limits (極限)

[5/11] L5 : Differentiation (連続性と微分法)

[5/18] L6 : Differentiation II : extrema, related rates ...（極値と...）

[5/25] L7 : Differentiation III : Newton's method, Taylor's expansion (ニュートン法とテイラー展開）

[6/1] L8 : Mid-term test. Integration I : definition, fundamental theorem of calculus　積分I.

[6/8] L9 : computation of indefinite integrals　不定積分

[6/15] L10 : Application of Integration I : length, volume and surfaces
積分の応用：長さ、面積、体積

[6/22] L11 : Application of Integration II : average, center of mass　（質量中心）~~, work of a force.~~

[6/29] L12 : Ordinary Differential equations (one variable)　常微分方程式

[7/6] L13 : Linear Differential Equations of order 2 : harmonic oscillators (small-angle pendulum, spring). 二階線形微分方程式：調和振動子（振幅が小さい振り子、ばね)

[7/13] L14 : Ordinary Differential Equations with Mathematica. Mathematicaを利用して常微分

ODE I: content

1. Introduction. Direction field. Examples
   入門。方向場。例

2. A simple case: linear differential equation of order 1 with constant coefficients
   定数係数の一階線形常微分方程式

3. A mini classification of Differential Equations
   微分方程式の簡単な分類

4. Method of integrating factor
   積分因子法

Ordinary Differential Equations (ODE)（常微分方程式）

- **Differential equations** are equations containing derivatives.（**導関数**を含む方程式。その解は関数である）

$$y'(t) = F(t, y(t))$$

t: time

Direction of the future state. 次の状態への方向

Present state parameter: y
現在の状態のパラメータ y

Function that expesses the future state in function of the present state.
現在の状態により、次の状態を表す。

A broad range of applications

- Examples of **physical phenomena** involving rates of change （変化率を関与する現象の例）

Engineering 技術・工学

  - Motion of fluids （流体運動）
  - Motion of mechanical systems（機械系の運動）
  - Flow of current in electrical circuits （電気回路における電流の流れ）
- **Biophysics**: membrane action potential of nerve cells (神経細胞の膜電位 Hodgkin-Huxley, Nobel Prize 1963)
- **Chemistry:** rate equation (反応速度式) $\frac{d[A]}{dt} = -k_1[A]$
- **Population/biological systems dynamics**（人口動態）: Lokta-Volterra predator-prey equation (捕食者-被食者の増減方程式）

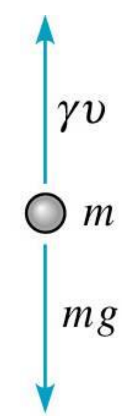
A differential equation that describes a physical process is often called a **mathematical model**.

Example 1: Free fall with air resistance (空気抵抗を含む自由落下)

Differential equation describing motion of an object falling in the atmosphere near sea level.
海面の近くの大気に落下している物体の運動を記述する微分方程式。

- Variables (変数): time t, velocity $\vec{v}(t)$ at time t.
- Newton's 2nd Law: $\sum \vec{F} = m\vec{a} = m(d\vec{v}/dt)$ ←total force
- Force of gravity(重力): $F_G = m\,g$ ←downward force
- Force of air resistance (空気抵抗の力): $\overrightarrow{F_R} = -\gamma\,\vec{v}$ ←upward
- Then

$$m\frac{dv}{dt} = mg - \gamma v$$



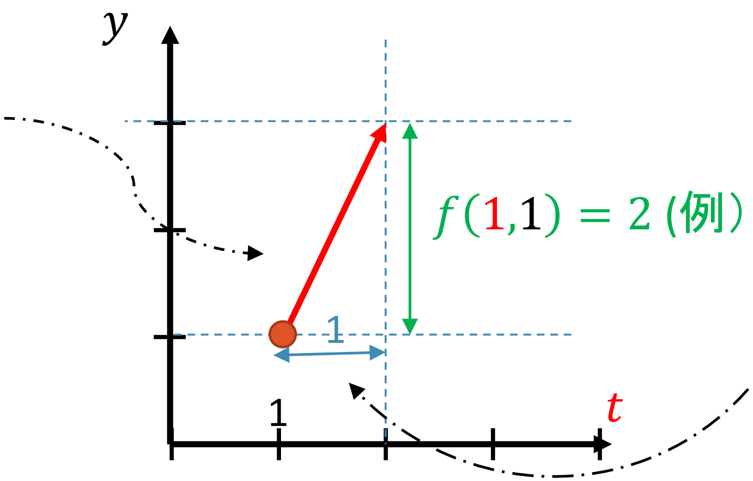
- Taking g = 9.8 m.sec^{-2}, m = 10 kg, γ = 2 kg.sec^{-1}, we obtain

$$\frac{dv}{dt} = 9.8 - 0.2v$$

Direction field (of 1st order diff. eqn.)
方向場　（一階線形微分方程式）

- $y' = f(t, y)$. The function of time $y: \subset E \to \mathbb{R}$ is viewed as a variable.
  $y' = f(t, y)$において、変数tによる関数　$y: E \subset \mathbb{R} \to \mathbb{R}$を変数を見なす。

Direction at the point $(t = 1, y = 1)$
点$(t = 1, y = 1)$における方向



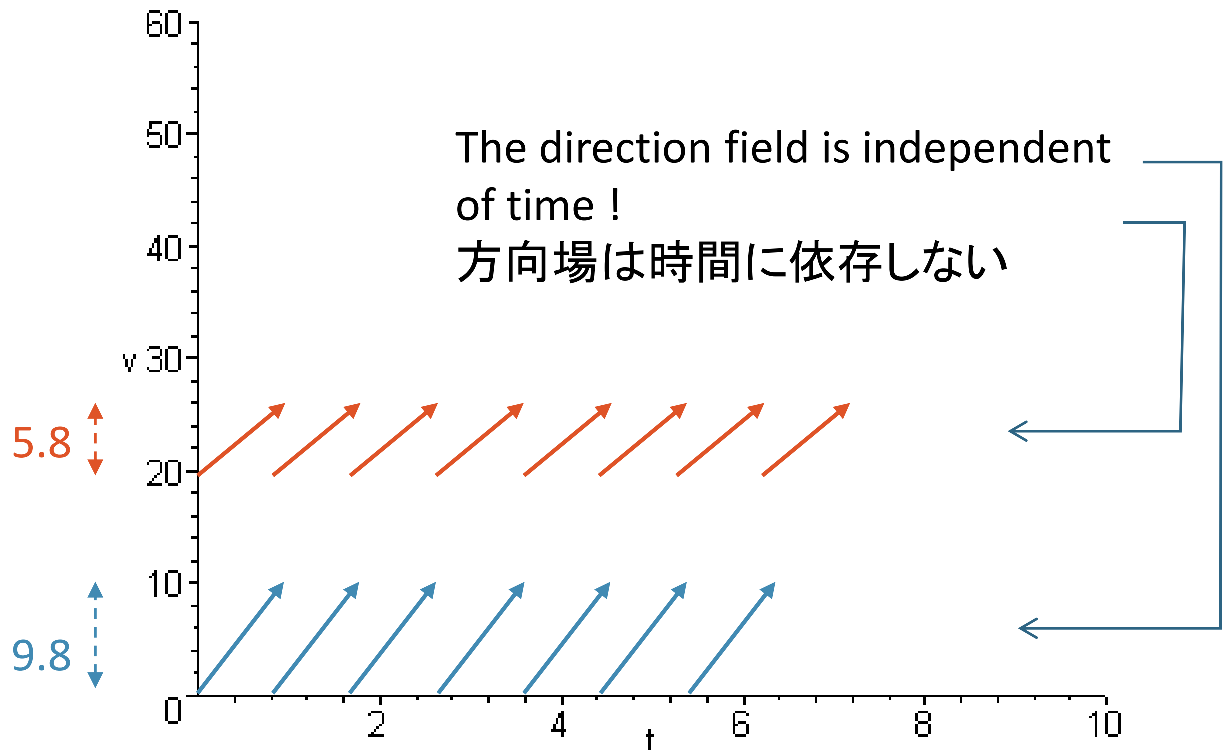
Move forward of 1 in the time direction
時間方向に1を進む

- The directions at all values (t, y) is called the direction field.
  各(t, y)の値に対して傾きからなるベクトル場を方向場という。

Ex.1: Direction field (or slope field)方向場

- For each value of time and speed t, v we can use the diff. eqn $v' = 9.8 - 0.2v$ to compute v'.
- The value v' is the slope of an arrow at coordinate (t, v)（v'の値は(t, v)座標にあるベクトルの傾きを表れる）

v	v'
0	9.8
5	8.8
10	7.8
15	6.8
20	5.8
25	4.8
30	3.8
35	2.8
40	1.8
45	0.8
50	-0.2
55	-1.2
60	-2.2



Ex.1: Direction field (or slope field)方向場

- To plot a direction field with **Mathematica**, use:
  `VectorPlot[{1,9.8-0.2v} , {t,0,10} , {v,0,80}]`

v	v'
0	9.8
5	8.8
10	7.8
15	6.8
20	5.8
25	4.8
30	3.8
35	2.8
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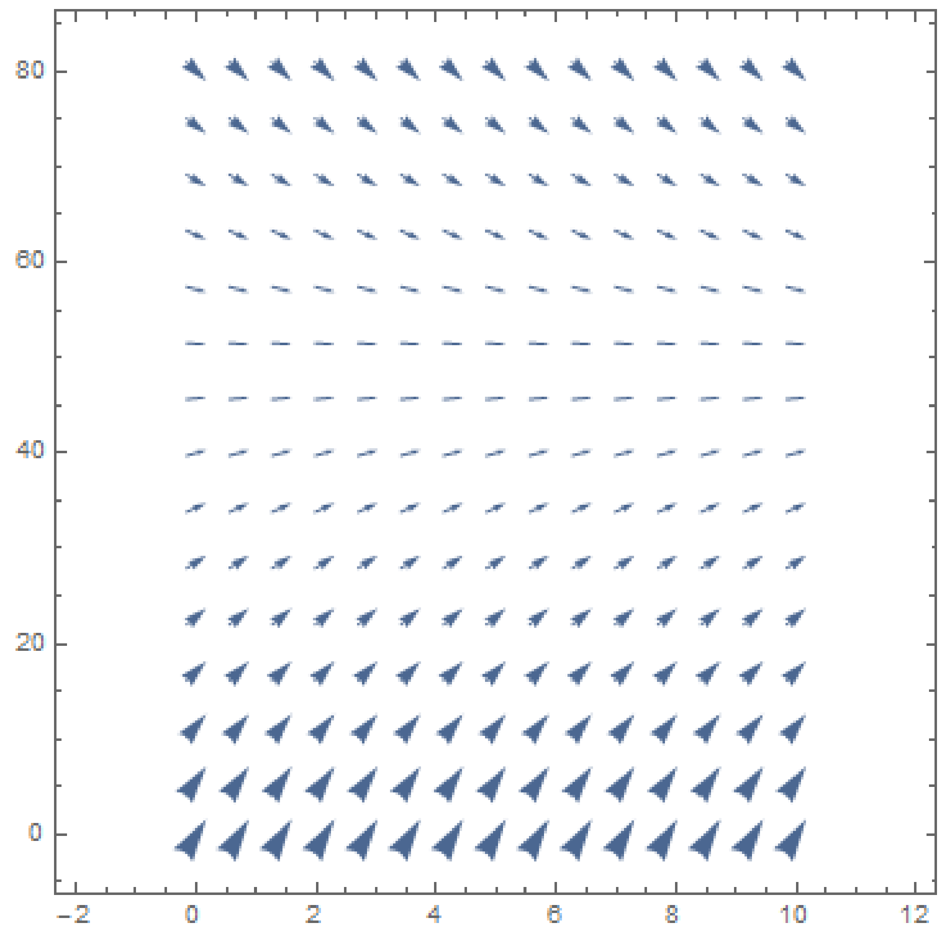
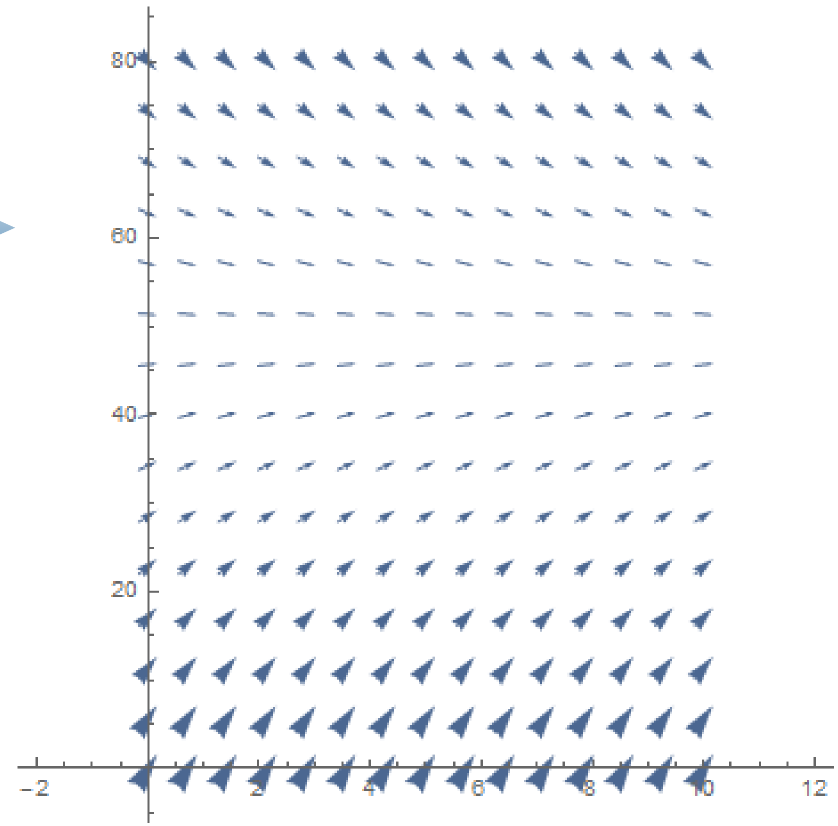


Ex.1: Direction field and equilibrium solution
方向場と均衡解（定数関数）

- Arrows are tangent lines to solution curves.
  矢印は解曲線に接線である。
- Horizontal solution $(y' = 0)$ curves are constant functions called equilibrium solutions

水平の解曲線$(y' = 0)$は定数関数で、
均衡解と呼ばれる。

Questions:



1. What is the equilibrium solution?　均衡解は何か？
2. How are the solutions above the equilibrium ?
   均衡の上に位置する解はどんなふるまいを表すか？
   And the solutions below ?
   下の解に対しては何か。

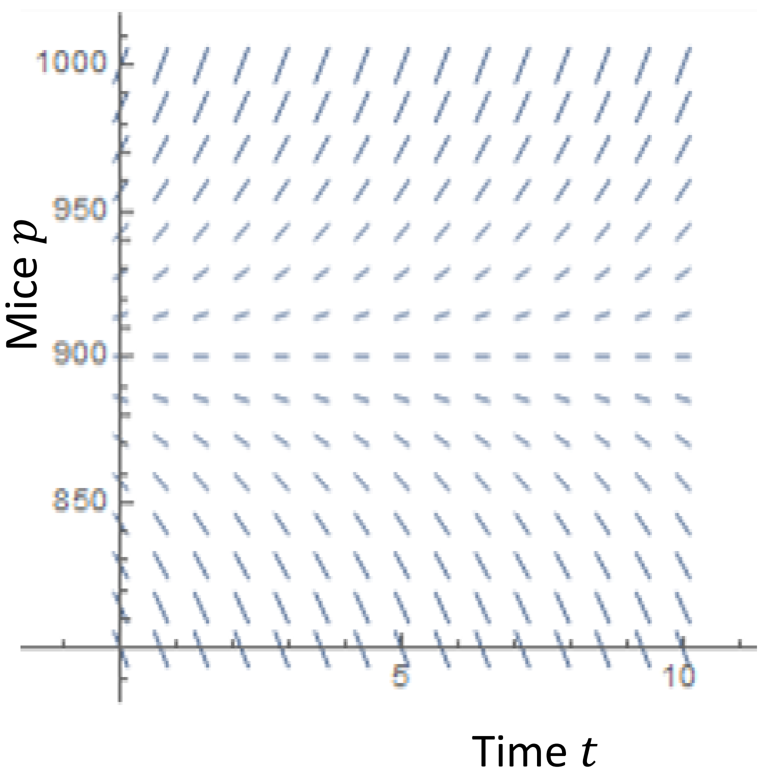
Example 2: Mice and Owls (ネズミとフクロウ)

- $p(t)$: population of mice at time t (時点tのネズミの人口)
- Reproduction rate: proportional to the current population (現在の人口に比例した)
  Rate(比例数): 0.5mice/month (without owl)
- If there are owls, they eat 15 mice/day
  フクロウがいると、日にネズミ15匹を食べる。
- **Question:** If 30 days=1 month, what is the differential equation that models the population $p(t)$ of mice?
  一か月は30日を想定したら、ネズミの人口$p(t)$をモデルする微分方程式は何か？

$$\frac{dp}{dt} = 0.5p - 450$$

Direction field of the equation

- $p(t)$: population of mice at time t
- **Question**: 1) what is the equilibrium solution ($p' = 0$)? 均衡解はなにか。



$$\frac{dp}{dt} = 0.5p - 450$$

2) What is the behavior of the solutions above the equilibrium？均衡の上に位置する解はどんなふるまいを表すか？

And below？ 下の解は？

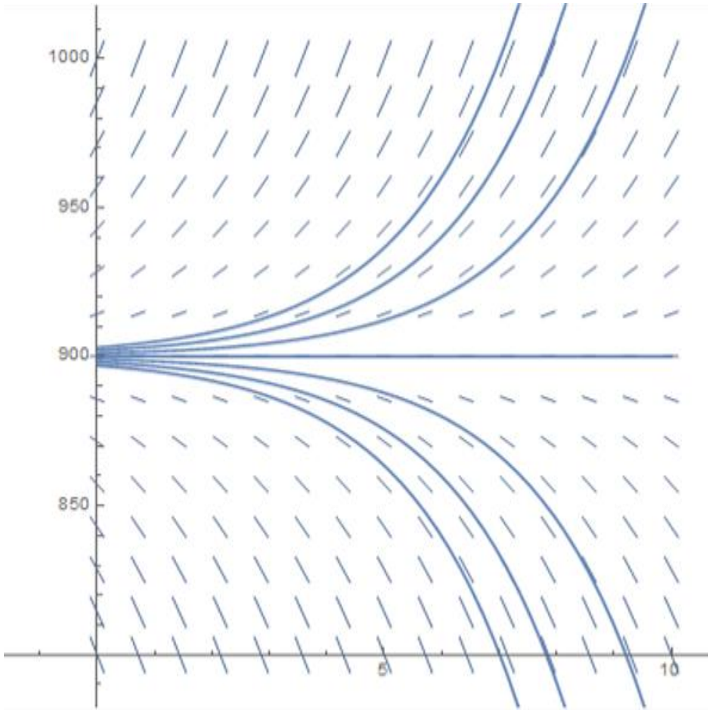
Solving the equation（方程式を解く）

- **Exercise:**
  Find solutions of the equation: $p' = 0.5p - 450.$

$$p = 900 + ke^{0.5\,t}$$, where k is constant.

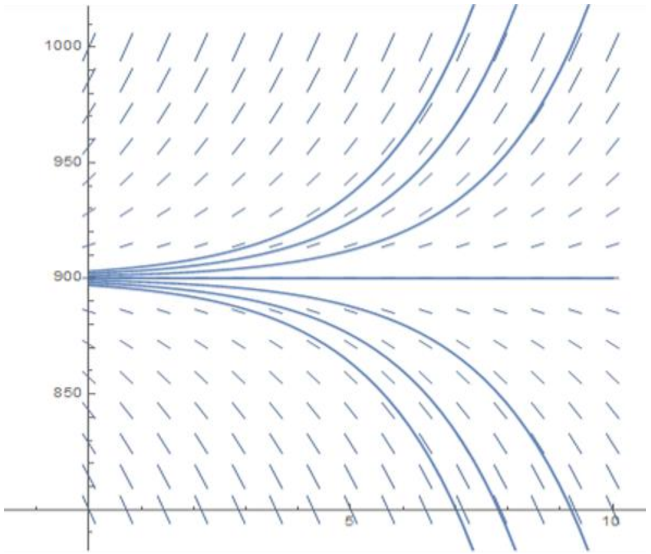
When $k = 0$, this is an equilibrium.

When $k \neq 0$, the solutions go away from the equilibrium with exponential decay.
解は指数関数的崩壊で均衡解から離ていく。



Solutions and initial value: 解と初期値

- A solution $\phi(t)$ verifies: $\phi'(t) = f\big(t, \phi(t)\big)$.
- On the direction field, this corresponds to solution curves:
  方向場で解曲線にあたる。
  - There are many curves !
    曲線がさまざまなある。
  - They never cross if f is continuous
    fは連続であれば、交差しない。
  - They never touch if $f(t, y)$ is smooth.
    f は「スムーズ」だったら、
    接触しない



Theorem: To each initial value y_0, there is one and only solution curve $y(t)$ such that $y(0) = y_0$.
各初期値y_0に対いて、$y(0) = y_0$を満たす解曲線$y(t)$が唯一在する。

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Solving 1st order linear differential equation. with constant coeff.

- Free fall and mice/owls differential equations
  $v' = 9.8 - 0.2v$ and $p' = 0.5p - 450$

  are of the form $y' = ay + b,\ a, b \in \mathbb{R}.$

1st derivative
第一階微分

- Order 1 linear diff. equation with constant coefficients
  定数係数の一階線形常微分方程式
- We can find exact solutions by using calculus.
  微分積分法を利用して厳密解を求められる。
- Otherwise…
  if we cannot find an exact solution（厳密解が無い）
  we can use a software to find an exact (sometimes….)
  or approximate solutions (most of the time).

1^{st} order Differential Equation
定数係数一階線形微分方程式

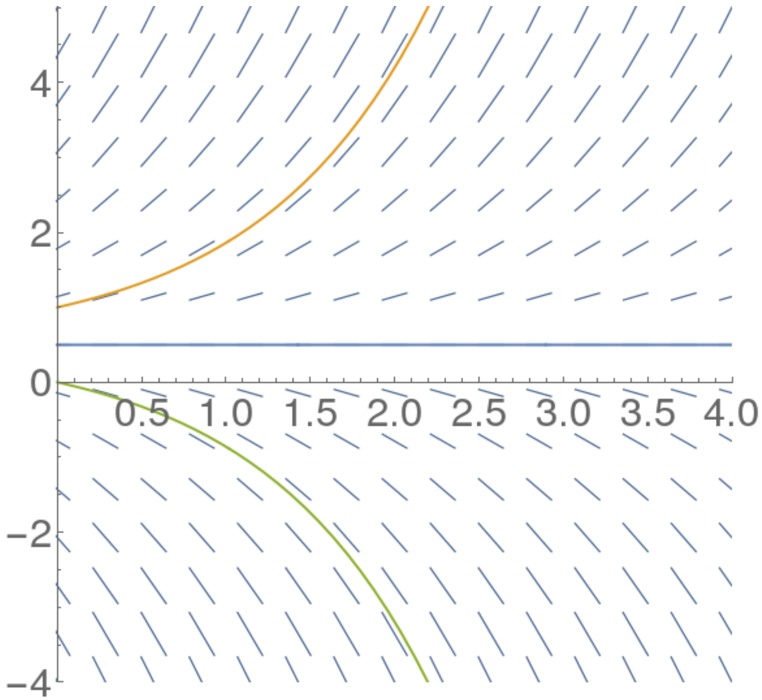
- Constant coefficients ($a, b \in \mathbb{R}, a \neq 0$)（定数係数）

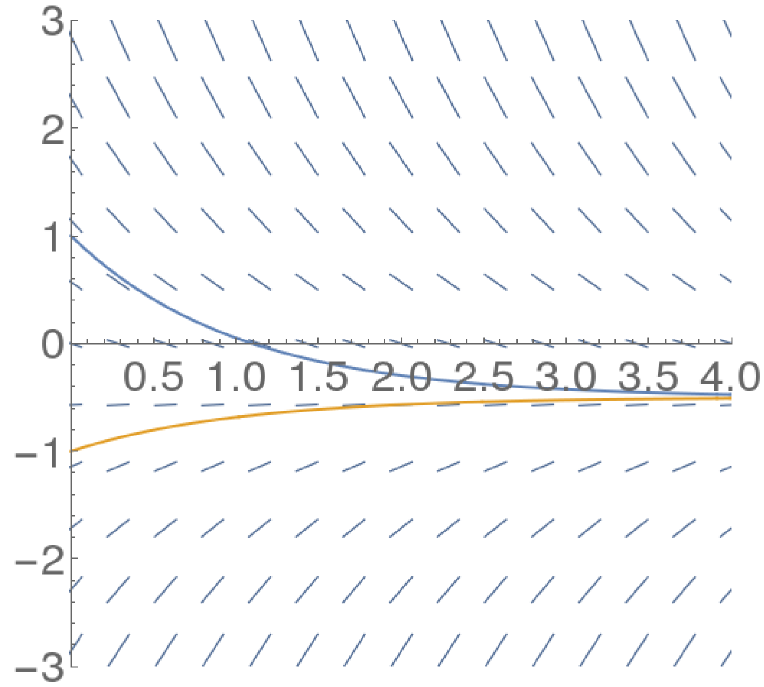
$$\begin{cases} y'(t) = ay(t) - b, \\ y(0) = y_0 \qquad \text{(initial value)} \end{cases}$$

Solution（解）:

$$y(t) = \frac{b}{a} + \left(y(0) - \frac{b}{a}\right)e^{at}$$

1. If $y_0 = b/a$, then y is constant, with $y(t) = b/a$ (equilibrium)
2. If $y_0 > b/a$ and $a > 0$, then y increases exponentially without bound.（上限なく指数関数的に増加する）
3. If $y_0 > b/a$ and $a < 0$, then y decays exponentially to b/a yはb/aに指数関数的に減衰する。
4. If $y_0 < b/a$ and $a > 0$, then y decreases exponentially without bound （下限がなく指数関数的に減少する）
5. If $y_0 < b/a$ and $a < 0$, then y increases asymptotically to b/a (yはb/aに漸近的に増加する。)





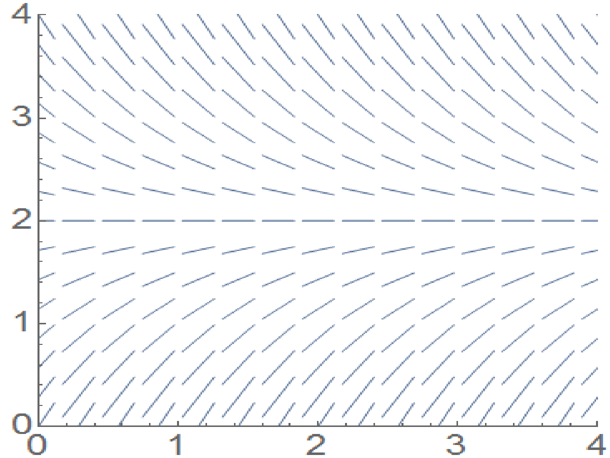
- Cases 1,2,4
- $a = 1 > 0, b = \frac{1}{2}$
- $y_0 = \frac{b}{a} = \frac{1}{2}, y_0 = 1,$ $y_0 = 0$

- Cases 3,5
- $a = -1 < 0, b = \frac{1}{2}$
- $y_0 = 1, y_0 = -1$

Exercise

-	A	$y' = 2y - 1$
2	B	$y' = y - 2$
6	C	$y' = y(y-3)$
1	D	$y' = 2 - y$
-	E	$y' = 1 - 2y$
4	F	$y' = 2 + y$
-	G	$y' = y(y+3)$
-	H	$y' = 1 + 2y$
6	I	$y' = y(3-y)$
3	J	$y' = -2 - y$

1



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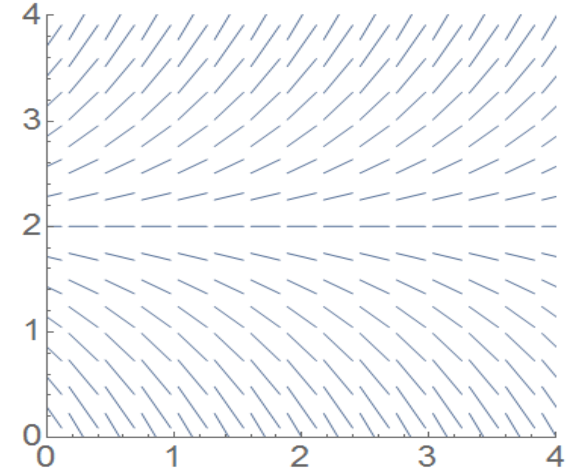

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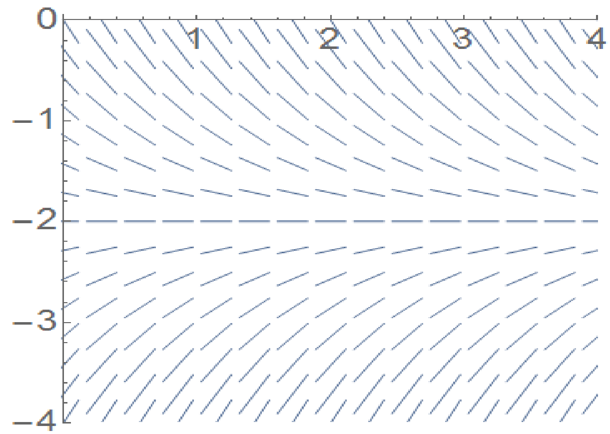

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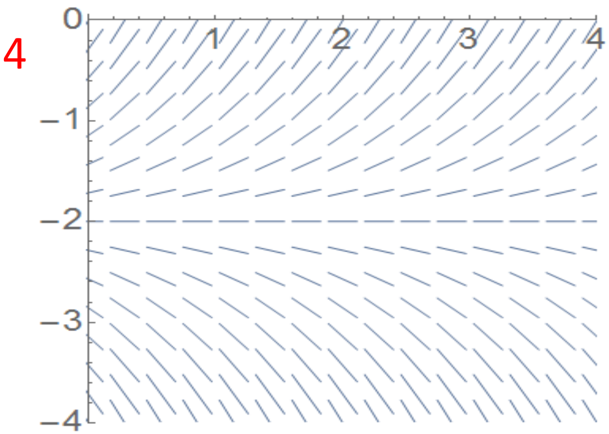

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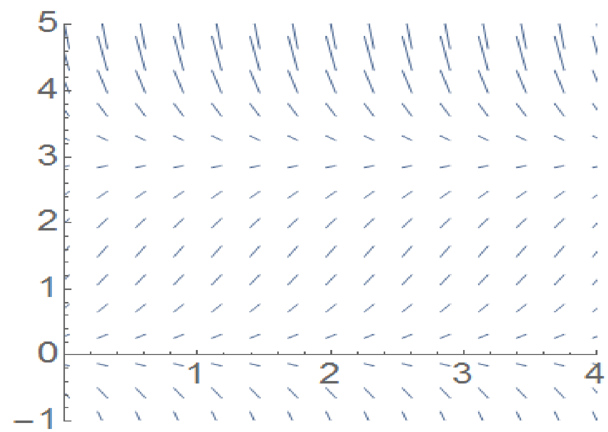

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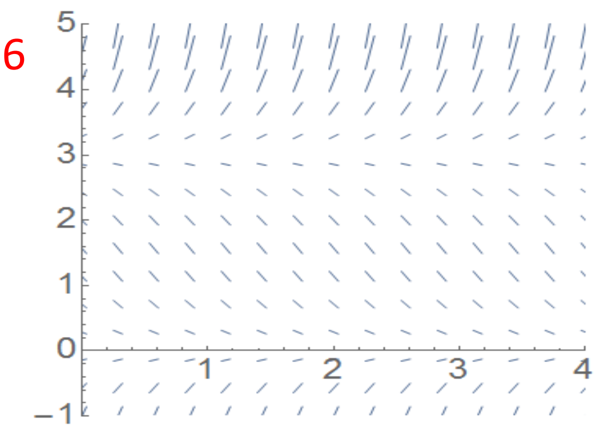

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A (mini) classification to Differential Equations

- **Ordinary Differential Equations (ODE)**
  常微分方程式
  The unknown function depends of a single variable.
  未知関数は一つ変数だけに依存するとき。
  Examples: Mice/Owl equation, free fall equation.

- **Partial differential Equation (PDE) 偏微分方程式**
  When the unknown function depends of several variables, and partial derivatives appear in the equation:
  未知関数はいくつかの変数に依存する、かつ　方程式で偏微分が現れる。
  Example: $\alpha^2 \frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial u(x,t)}{\partial t}$ (heat equation)

Systems of differential equations
微分方程式系 (studied in Essential Math. II in October)

- Number of unknown functions >1（未知関数の個数は>1）
- **Example**: predator-prey equations (like mice/owl but when owls population $q(t)$ is considered also)
  補食者ー被食者系 （ネズミ・フクロウのようなものだが、ここでフクロウの個数 $q(t)$も考慮され、未知関数がふたつある）

1. Slide 11→ Reproduction rate of mouse without owl: 0.5mice/month $p'(t) = 0.5p(t)$
2. If there are no mouse, the owl population dies out by 70% in a month: $q'(t) = -0.7q(t)$
3. When owls meet mouse: owl population increases by a factor $0.2q(t)p(t)$ and the mouse population decreases by a factor $0.3q(t)p(t)$

$$\left.\begin{aligned} p'(t) &= 0.5p(t) - 0.3q(t)p(t) \\ q'(t) &= -0.7q(t) + 0.2q(t)p(t) \end{aligned}\right\}$$

System of 2 differential equations in 2 unknowns p and q

Order of a differential Equation. Linear equation.

- Order of the highest derivative that appears in the equation.　最高階導関数の階数がn次である場合、その微分方程式をn階微分方程式と呼ぶ。

注: We will study only diff. equations where the highest derivative can be isolated (and max $n = 2$)

$$\frac{d^n y}{dt^n} = f\left(t, y, \frac{dy}{dt}, \frac{d^2 y}{dt^2}, \dots, \frac{d^{n-1} y}{dt^{n-1}}\right)$$

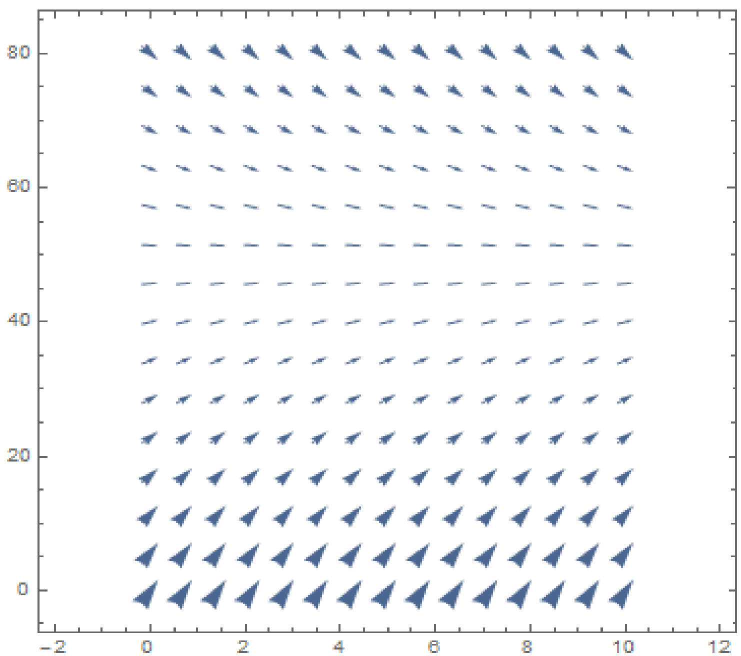
Order n linear differential equation n階線形微分方程式

$$a_0(t)\frac{d^n y}{dt^n} + a_1(t)\frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_n(t)y = g(t)$$

tによる関数だが、yが現らない

Autonomous ODE (自律的なODE)

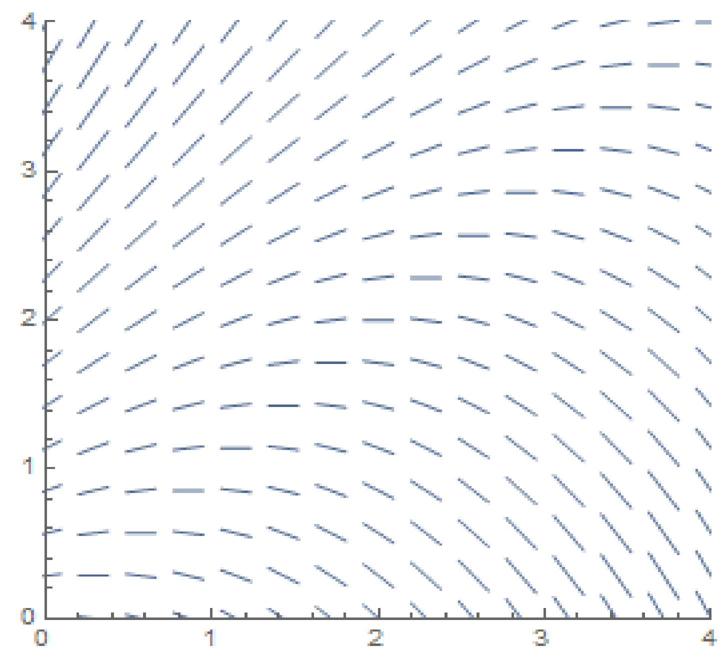
- $y' = f(t, y)$ is an ODE.
- If f is independent of t then the equation is autonomous. (fは時間に依らないとき、ODEは自律的という)
- Solution curves are then "time-invariant"



AUTONOMOUS

$y' = 9.8 - 0.2y$

NON-AUTONOMOUS



$y' = y - t$

	Order ?	Autonomous?	Linear?	Constant coefficients?
$y' = 3y + 2$	1	○	○	○
$\frac{d^4y}{dt^4} + t\frac{d^3y}{dt^3} - 3\frac{d^2y}{dt^2} + \frac{dy}{dt} = 0$	4	×	○	×
$y' = 3y$	1	○	○	○
$\theta'' + 10\,sin(\theta) = 0$	2	○	×	—
$\frac{dy}{dt} + ty^2 = 0$	1	×	×	—
$\frac{d^3y}{dt^3} + t\frac{dy}{dt} + (cos(t))^2 y = t^3$	3	×	○	×
$t^2\frac{d^2y}{dt^2} + t\frac{dy}{dt} + 2y = sin(t)$	2	×	○	×

Example of order 2 non-linear ODE

- Pendulum! There is no resistance nor friction.
  (振り子：空気抵抗も摩擦もないという仮定)
- The position of the pendulum is completely given by the angle θ
  （振り子の位置が角度θに完全に与えられる)
- By Newton's law of mechanics we get:
  ニュートンの力学の原理によって：

Angle 角度

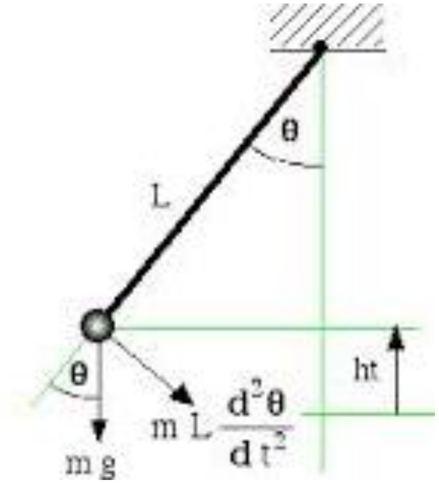
Gravitational constant 万有引力定数

Length of the pendulum 振り子の長さ

$$mg\sin(\theta) + mL\frac{d^2\theta}{dt^2} = 0$$

$$\frac{d^2\theta}{dt^2} = -\frac{g}{L}\sin(\theta)$$

Does not depend of the mass m
質量mに依存しない



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1st order linear ODE: Method of integrating factors（積分因子法）

- Page 17 → How to solve 1st order linear ODE with constant coefficient （定数係数の1階線形常微分方程式）
- And if the coefficients are non-constant ? (⇔ the equation is not autonomous) ?
  係数は定数でない時に？(⇔自律的でない)
- Example: $y' + 1/2\, y = 1/2\, e^{\frac{t}{3}}$ → not constant
- Multiplying both sides by a function $\mu(t)$

$$\mu(t)y' + 1/2\, \mu(t)y = 1/2\, e^{\frac{t}{3}}\mu(t)$$

- We want to choose $\mu(t)$ so that:

$$(\mu(t)y)' = \mu(t)y' + \mu'(t)y = \mu(t)y' + 1/2\, \mu(t)y$$

- Choose $\mu(t)$ so that

$$\mu'(t)y = \frac{1}{2}\mu(t)y \Rightarrow \mu(t) = e^{\frac{t}{2}}$$

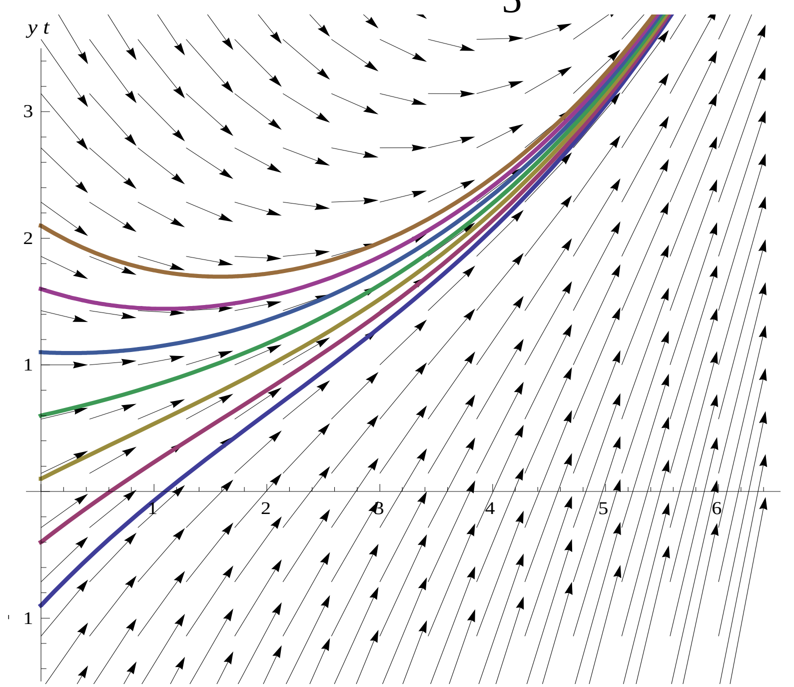
Method of integrating factors: Example (II)

- $\mu(t) = e^{t/2}$ is the integral factor function.
  (積分因子関数）
- we solve the original equation as follows:
  元の微分方程式を次の通りに解く:

$\times\,\mu(t)$

$$y' + 1/2\,y = 1/2\,e^{\frac{t}{3}}$$
$$e^{\frac{t}{2}}\,y' + 1/2\,e^{\frac{t}{2}}y = 1/2 \cdot e^{\frac{5t}{6}}$$
$$\left(e^{\frac{t}{2}}y\right)' = 1/2 \cdot e^{\frac{5t}{6}}$$
$$e^{\frac{t}{2}}y = 3/5 \cdot e^{\frac{5t}{6}} + C$$

Sample Solutions : $y = \frac{3}{5}e^{t/3} + Ce^{-t/2}$



- General Solution (一般解）

$$y = \frac{3}{5}e^{\frac{t}{3}} + Ce^{-\frac{t}{2}}$$

Integrating factor: General Case

- Next, we consider a general first order linear equation
  次に、一般な一階線形常微分方程式を考える:

$$y' + p(t)y = g(t)$$

- Multiplying both sides by $\mu(t)$, we obtain
  等式の左右に$\mu(t)$を乗じると、以下を得る:

$$\mu(t)\frac{dy}{dt} + p(t)\mu(t)y = g(t)\mu(t)$$

- Next, we want $\mu(t)$ such that $\mu'(t) = p(t)\mu(t)$:
  $\mu'(t) = p(t)\mu(t)$ が成り立つように$\mu(t)$を求める:

$$\frac{d}{dt}[\mu(t)y] = \mu(t)\frac{dy}{dt} + p(t)\mu(t)y$$

- Thus we want to choose $\mu(t)$ such that $\mu'(t) = p(t)\mu(t)$.

Integrating Factor: General Case (II)

- Thus we want to choose $\mu(t)$ such that $\mu'(t) = p(t)\mu(t)$.
  ゆえに、$\mu'(t) = p(t)\mu(t)$ を満たす積分因子関数$\mu(t)$をとりたい：
- Assuming $\mu(t) > 0$, it follows that

$$\int \frac{d\mu(t)}{dt} = \int p(t)dt \Rightarrow \ln \mu(t) = \int p(t)dt + k$$

- Choosing $k = 0$, we then have

$$\boxed{\mu(t) = e^{\int p(t)dt}}$$

  and note that $\mu(t) > 0$ as desired.

- 注：Must be able to compute $\int p(t)\, dt$

Homework (July 13th)

- Solve the differential equation by the integrating factors method

$$ty' + 2y = 4t^2$$

With the initial condition $y(1) = 2$.

- (Use the file “ODE-Mathematica.nb”)
  - Plot the direction field of the equation.
  - Use DSolve to solve the equation
  - Plot on the same graph the direction fields and the solution