Essential Mathematics for Global Leaders I

Lecture 5-3

Integration III

2015 June 29th

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Plan (tentative)

[4/13] L1 : introduction. Review of high-school mathematics in English.

[4/20-27] L2-3 : Functions and graphs. Plotting with Mathematica I (グラフをプロットする)

[5/7] L4 : Infinitely small and large : limits (極限)

[5/11] L5 : Differentiation (連続性と微分法)

[5/18] L6 : Differentiation II : extrema, related rates ...（極値と...）

[5/25] L7 : Differentiation III : Newton's method, Taylor's expansion (ニュートン法とテイラー展開）

[6/1] L8 : Mid-term test. Integration I : definition, fundamental theorem of calculus　積分I.

[6/8] L9 : computation of indefinite integrals　不定積分

[6/15] L10 : Application of Integration I : length, volume and surfaces
積分の応用：長さ、面積、体積

[6/22] L11 : Application of Integration II : average, center of mass （質量中心）~~, work of a force.~~

[6/29] L12 : Ordinary Differential equations (one variable)　常微分方程式

[7/6] L13 : Linear Differential Equations of order 2 : harmonic oscillators (small-angle pendulum, spring). 二階線形微分方程式：調和振動子 (振幅が小さい振り子、ばね)

[7/13] L14 : Ordinary Differential Equations with Mathematica. Mathematicaを利用して常微分

Integration III: content

1. Example of application 1 : Average and pressure 応用2: 平均化と圧力

2. Example of application 2: Center of mass 応用2 : 質量中心

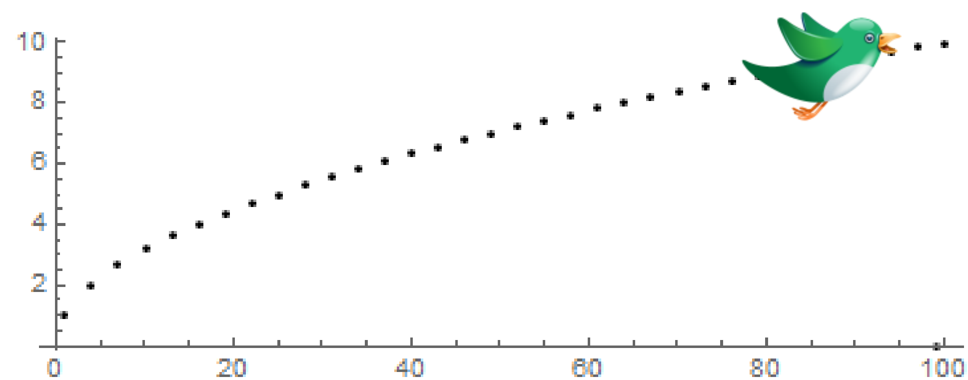
Integral as average
積分は平均をとる手法をみなす

- **Example:** Average height of a bird during a flight.
  (鳥の平均高度飛行）
  A bird is flying along the curve $y = \sqrt{x}$ from the value at the ground $x = 0$ and $x = 10$.
  鳥は地面における値$x = 0$から$x = 10$までの曲線$y = \sqrt{x}$に沿って飛んでいる。

Question:

What is the average height of the bird ?

Answer: $\frac{1}{10}\int_0^{10} \sqrt{x}\, dx$

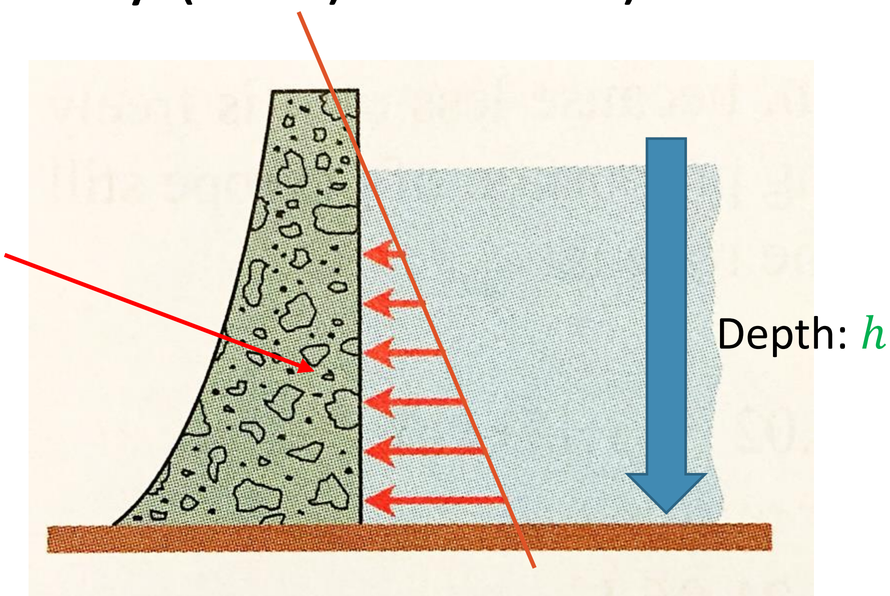


Theorem: The **average** value of a continuous function $y = f(x)$ between $x = a$ and $x = b$ is $\frac{1}{b-a}\int_a^b f(x)\, dx$

Fluid pressure and force I
流体の圧力と力

- Pressure p $(N.m^{-2})$ at depth h is : $p = wh$
  (w is weight-density (密度) $N.m^{-3}$)

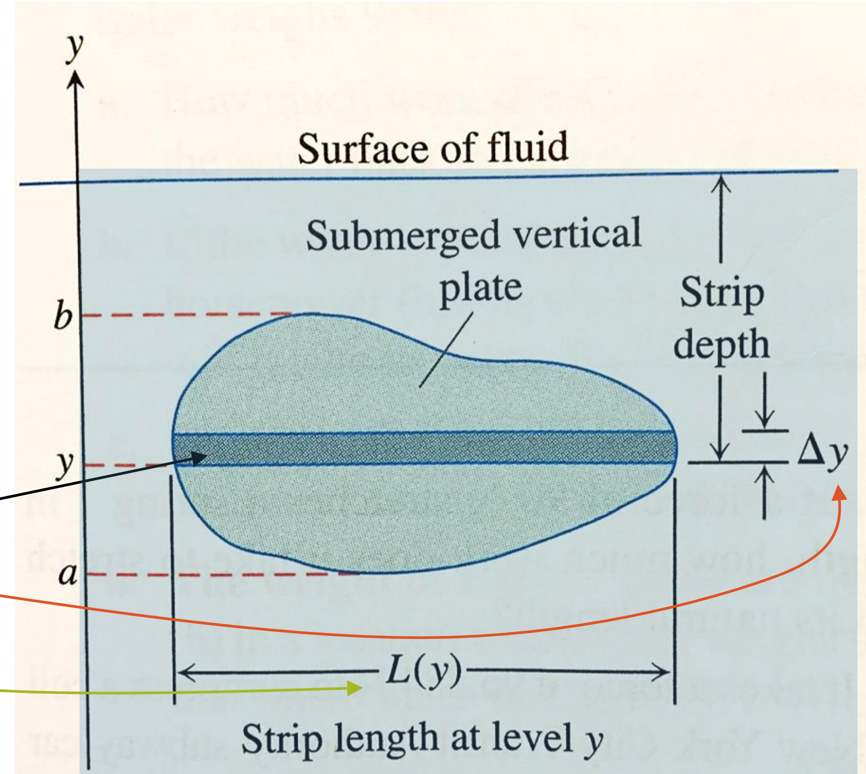
Pressure is increasing linearly with depth
圧力は深さと比例する



- Force exerted on a surface of area A by a constant pressure:
  面積Aの外面に定圧によって加えられる力

$$F = total\ force = force\ per\ m^2 \times area$$
$$= pressure \times area = pA = whA = F$$

Fluid pressure and forces II



- Integral for fluid force against a vertical flat plate:
- ΔF =(pressure along bottom edge)x(area) $= w(\text{strip depth})L(y)\Delta y$
- Cut the plate into n strips:

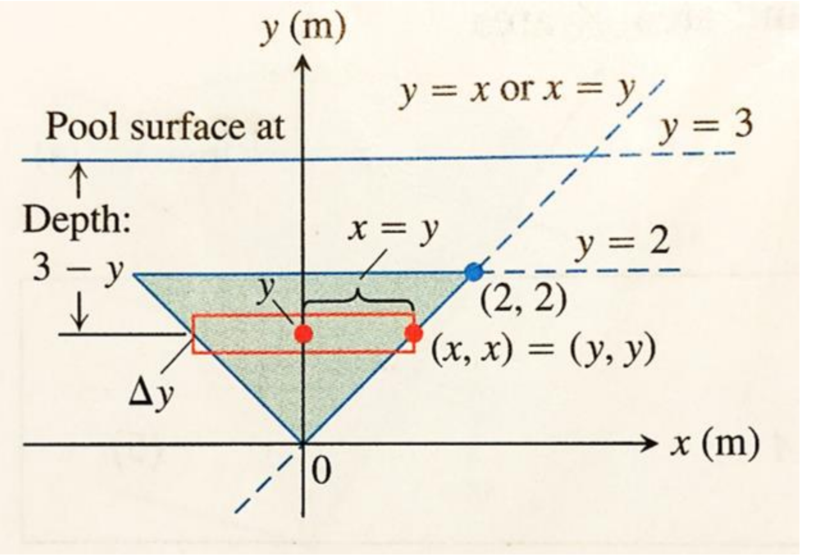
$$F \approx \sum_{k=1}^{n} \left(w \cdot (strip\ depth)_k \cdot L(y_k)\right) \Delta y_k$$

- When $n \to \infty$,

$$\lim_{n\to\infty} \sum_{k=1}^{n} (w \cdot \cdots)\Delta y_k = \int_a^b w \cdot (strip\ depth) \cdot L(y)\, dy = F$$

(F: force exerted by the a fluid pressure against one side of the plate)
平板の一面に流体の圧力によって加えられる力）

Exercise:



- A triangle vertical plate is underwater in a pool. Compute the force exerted by the water on the surface of the plate. (プールの水面下に垂直三角形の平板がある。平板の一面に流体によって加えられる力の全体を計算せよ）。

Integration III: content

1. Example of application 1 : Average and pressure 応用2: 平均化と圧力

2. Example of application 2: Center of mass 応用2：質量中心

Moments and centers of mass
第一次モーメントと質量中心

- Masses along a line: moment of m_i is $m_i x_i$

- Moment of the system about the origin（原点）is:

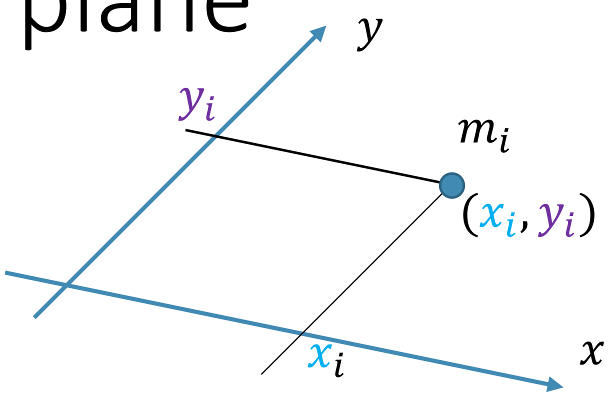
$$m_1 x_1 + m_2 x_2 + m_3 x_3$$

- Find $\bar{x}$ such that the moment about $\bar{x}$ is 0:

$$m_1(x_1 - \bar{x}) + m_2(x_2 - \bar{x}) + m_3(x_3 - \bar{x}) = 0$$

- Solution: $\bar{x} = \dfrac{m_1 x_1 + m_2 x_2 + m_3 x_3}{m_1 + m_2 + m_3}$ (center of mass)

Masses situated in the plane
平面に位置する質量

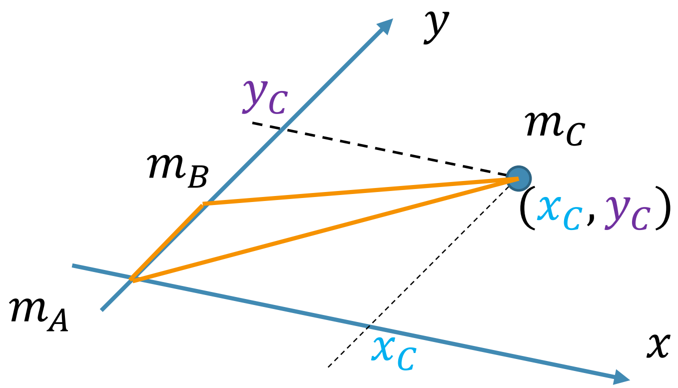


- Each mass m_i has two moments: $m_i x_i$ and $m_i y_i$
- Let M be the system total mass: $M = \sum_i m_i$
- Moment about x-axis (x-軸による第一次モーメント): $M_x = \sum_i m_i x_i$
- Moment about y-axis: $M_y = \sum_j m_j y_j$

$$\bar{x} = M_x / M, \qquad \bar{y} = M_y / M$$

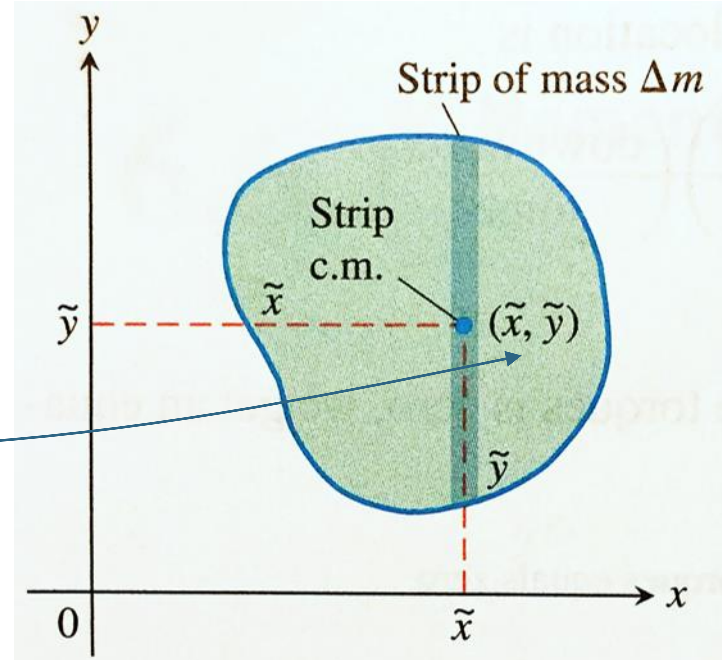
- The center of mass (質量中心) is the point $(\bar{x}, \bar{y})$

Exercise:



- What is the center of mass of 3 points ABC in general position, when $m_A = m_B = m_C$?

Center of mass of a region in the plane
面の質量中心



- Consider a thin, flat object of density δ
  密度 δ の細く平らな物体を考える。
- Let $(\tilde{x}, \tilde{y})$ be the center of mass of a strip of mass Δm
  （長細いの質量中心）
- Moments of the strip about the x-axis and y-axis:

$$\tilde{y}\Delta m \quad , \qquad \tilde{x}\Delta m$$

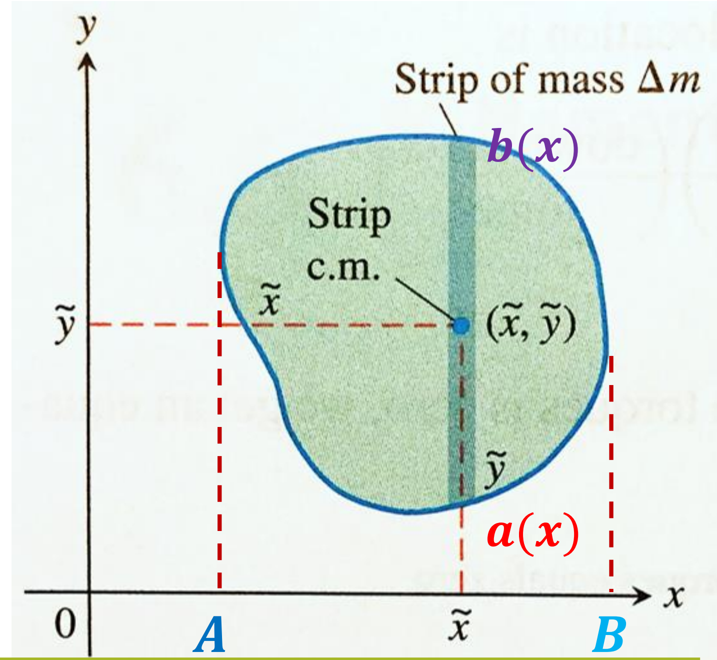
- Summing over the strips:

$$\bar{x} = \frac{M_x}{M} = \frac{\sum \tilde{x}\Delta m}{\sum \Delta m} \quad , \qquad \bar{y} = \frac{M_y}{M} = \frac{\sum \tilde{y}\Delta m}{\sum \Delta m}$$

- When number of strips is going to ∞:

$$\bar{x} = \int \tilde{x} dm / M \quad , \quad \bar{y} = \int \tilde{y} dm / M$$

Center of mass: average of mass over a surface along each axes



Let $\tilde{x}, \tilde{y}$ be the center of mass of the strip of density δ:

- Density at point (x, y): $\delta(x, y)$
- Width: dx
- Mass of the strip dm:

  $$dm = \left(\int_{a(x)}^{b(x)} \delta(x, y) dy\right) \cdot dx$$

- $\tilde{x} = x$
- ~~$\tilde{y} = \int_{a(x)}^{b(x)} y \cdot \delta(x, y) dy$~~

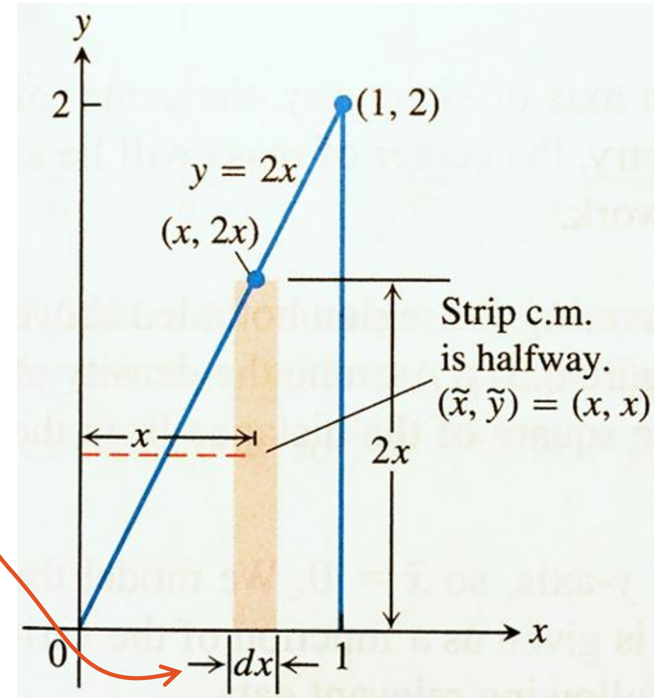
$$\tilde{y} = \int_{a(x)}^{b(x)} y \cdot \delta(x, y)\, dy \Big/ \int_{a(x)}^{b(x)} \delta(x, y)\, dy$$

- $\bar{y} = \int_A^B \tilde{y}\, dm \Big/ \int_A^B dm$ = "relative mass along the y-axis / total mass" = Average y-position
- $\bar{x} = \int_A^B \tilde{x}\, dm \Big/ \int_A^B dm$ = Average x-position

} In term of mass

Example: center of mass of a flat triangle (例 : 平らな三角形の質量中心）

- **Problem**: find the center of mass of the triangle of density $\delta = 3$ (in $kg.m^{-2}$)
- Consider a strip（長細い） of width dx.
  Mass of the strip: $dm = \left(\int_0^x \delta dy\right)dx \ = 6x \cdot dx$
  Center of mass: $\tilde{x} = x$
  $\tilde{y} = \int_0^{2x} y\delta\, dy/6x = 3[y^2/2]_0^{2x}/6x = x$



- Moment of the strip about the y-axis: $M_y = \int \tilde{x} dm$
- $M_y = \int x \cdot 6x\, dx = \int 6x^2 dx = [2x^3]_0^1 = 2$ kg.m
- Mass of the triangle: $\int dm = \int_0^1 6x\, dx = [3x^2]_0^1 = 3$ kg
- $\bar{y} = M_y/M = 2/3\ meter$

Exercise / Homework

1. Compute $\bar{x}$ the x-coordinate of the center of mass in the triangle page 14.

2. Find the center of mass of a thin plate covering the region bounded above the parabola $y = 4 - x^2$ and below the x-axis.
   The density of the plate at the point (x, y) is $\delta(x, y) = 2x^2$.