Essential Mathematics for Global Leaders I

Lecture 5-2

Integration II

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Plan (tentative)

[4/13] L1 : introduction. Review of high-school mathematics in English.

[4/20-27] L2-3 : Functions and graphs. Plotting with Mathematica I (グラフをプロットする)

[5/7] L4 : Infinitely small and large : limits (極限)

[5/11] L5 : Differentiation (連続性と微分法)

[5/18] L6 : Differentiation II : extrema, related rates ...（極値と...）

[5/25] L7 : Differentiation III : Newton’s method, Taylor’s expansion
(ニュートン法とテイラー展開）

[6/1] L8 : Mid-term test. Integration I : definition, fundamental theorem of calculus　積分I.

[6/8] L9 : computation of indefinite integrals　不定積分

[6/15] L10 : Application of Integration I : length, volume and surfaces
積分の応用:長さ、面積、体積

[6/22] L11 : Application of Integration II : average, center of mass （質量中心）, work of a force.

[6/29] L12 : Ordinary Differential equations (one variable)　常微分方程式

[7/6] L13 : Linear Differential Equations of order 2 : harmonic oscillators (small-angle pendulum, spring). 二階線形微分方程式:調和振動子 (振幅が小さい振り子、ばね)

[7/13] L14 : Ordinary Differential Equations with Mathematica. Mathematicaを利用して常微分

Integration II: content

1. Computation of primitives 2 (antiderivative)
   原始関数をとる（不定積分）

2. Methods to compute volumes
   体積を測定する方法

3. Length of curves and area of surfaces of revolution
   曲線の長さと回転面の面積

Computation of indefinite integral (= antiderivatives)

- **General techniques (一般な方法）**
  - Substitution rule (chain rule backward)（置換律）
  - Integration by parts （部分積分）

Previous lesson L5-1

- **Specialized techniques （特殊な問題向けの方法）**
  - Introduce new function: inverse trigonometric functions 新しい関数を導入する: 逆三角形関数
  - Rational Functions:（有理関数）
  - Rational Functions of trigonometric functions （三角形関数の有理関数）
  - ....
- **Automatic procedure (algorithm) (自動積分法）**
  - Liouville (1860), Risch (1960).

Not studied 紹介されていない

Specialized technique of antiderivation (I)
Inverse trigonometric functions (逆三角形関数)

- Example 1:

$$\int \frac{dx}{x^2+1} = Arctan(x) + C$$

- Example 2:

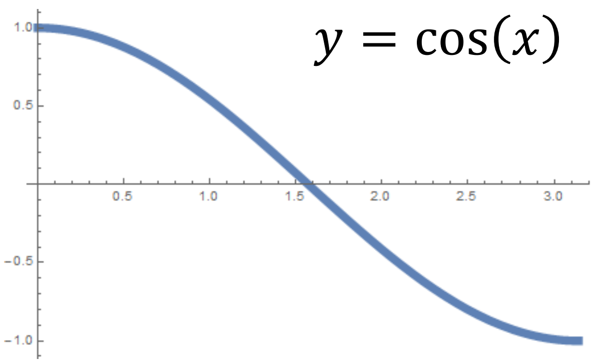
$$\int \frac{dx}{\sqrt{1-x^2}} = Arcsin(x) + C$$

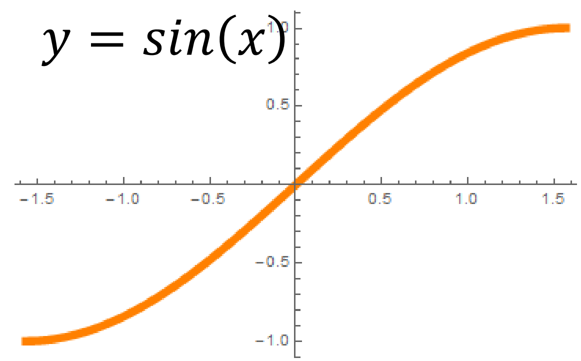
- Example 3: (looks like Example 2 but is very different)

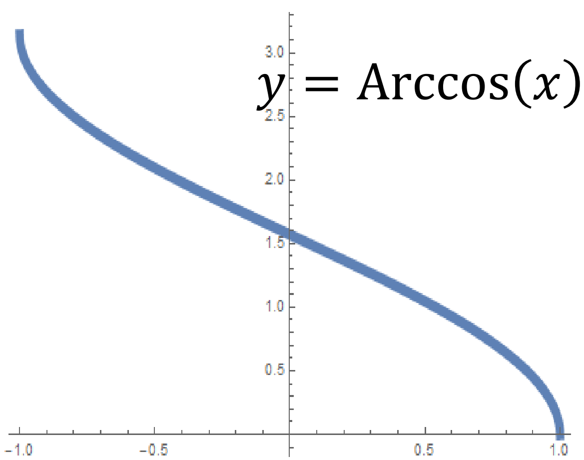
$$\int \frac{dx}{\sqrt{x^2-1}} = Log\left(x + \sqrt{x^2-1}\right) + C$$

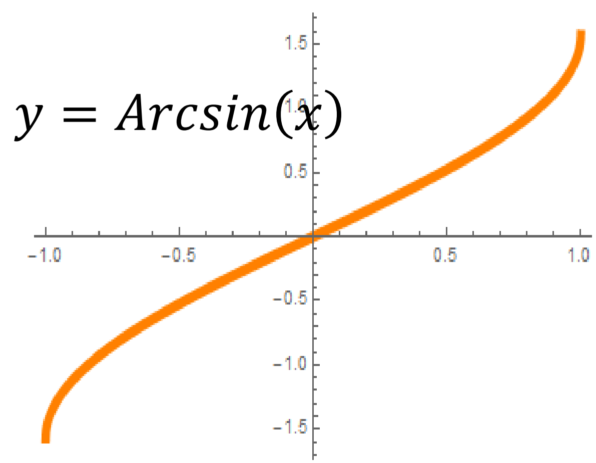
Inverse trigonometric functions (II)
(逆三角形関数)

- Graphs of $Arccos$ and $Arcsin$



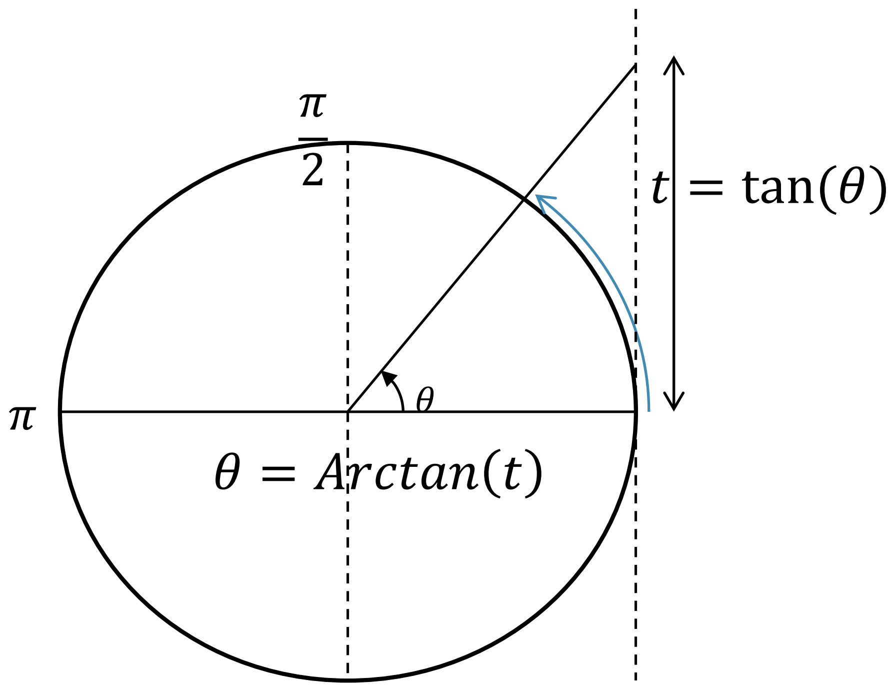


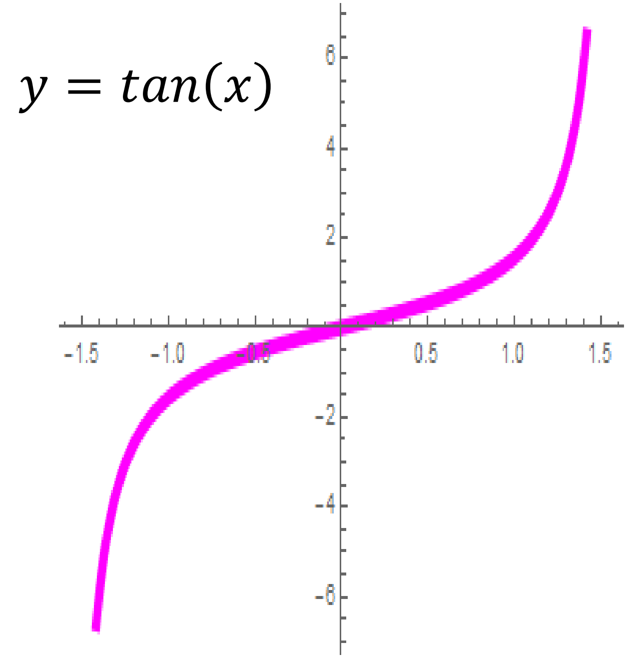


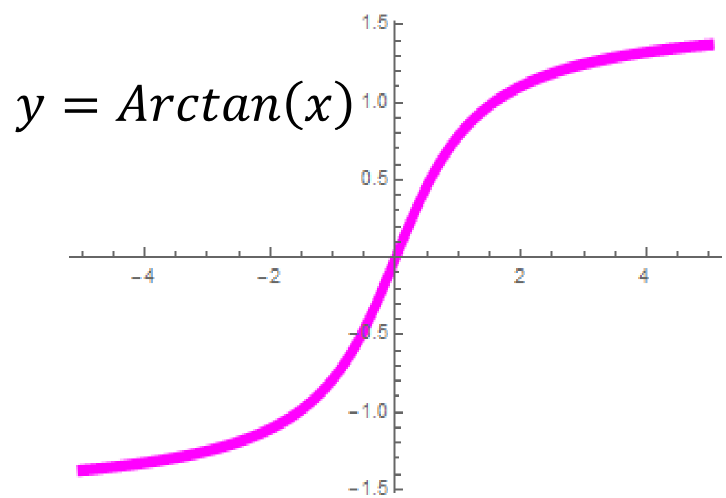


Arc Tangent

- $Arctan : \mathbb{R} \rightarrow \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$







Derivative of inverse trigonometric functions

- General property of inverse functions:
$$f \circ f^{-1}(x) = x$$
- Let $g(x) = f^{-1}(x)$.

- **<u>Exercise:</u>**

1. Differentiate the relation: $f \circ g(x) = x$
2. Deduce the derivative $g' = (f^{-1})'$
3. Deduce $(Arccos)'$
   $(Arcsin)'$
   $(Arctan)'$

Specialized techniques II:

- Rational functions:
$$f(x) = \frac{a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0}{b_m x^m + \cdots + b_1 x + b_0} = \frac{A(x)}{B(x)}$$

- Trigonometric polynomials and rational functions:
$f(x) = \cos(x)^3 \sin(x)^2 - 3\cos(x)\sin(x)^2$

- Others like integration tables …

→ always can be integrated

…Not very funny !

Automatic procedure to compute indefinite integrals (不定義積分をとるための基本手順)

- **Elementary functions:** （初等関数）Ln, Exp, Cos, Tan, Sin and their inverses: Arccos, Arcsin, Arctan etc…

- **Liouville's Problem** (in differential algebra 微分代数において) :
  Identify functions whose antiderative can be written as an elementary functions ?
  （初等関数によってとれる不定積分を持つ関数を識別すること）

- **Risch algorithm** (1968): solves the Liouville's problem.
  Difficult and long algorithm
  Implemented (実装された） in some maths software.

Automatic computation of antiderivative

- From Liouville's theorem (1840-50). Non-integrable functions: e^{-x^2}, $\frac{\sin(x)}{x}$, $e^{xln(x)} = x^x$, $\frac{1}{\ln(x)}$ have no antiderivatives in terms of elementary functions (初等関数によって表現できない不定積分を持つ関数）

- Create new functions！ (それでは、新たな関数を定義しよう）

$$Si(z) = \int_0^z \frac{\sin(t)}{t} dt, \qquad \mathrm{erf}(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-t^2} dt$$

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   原始関数をとる (不定積分）

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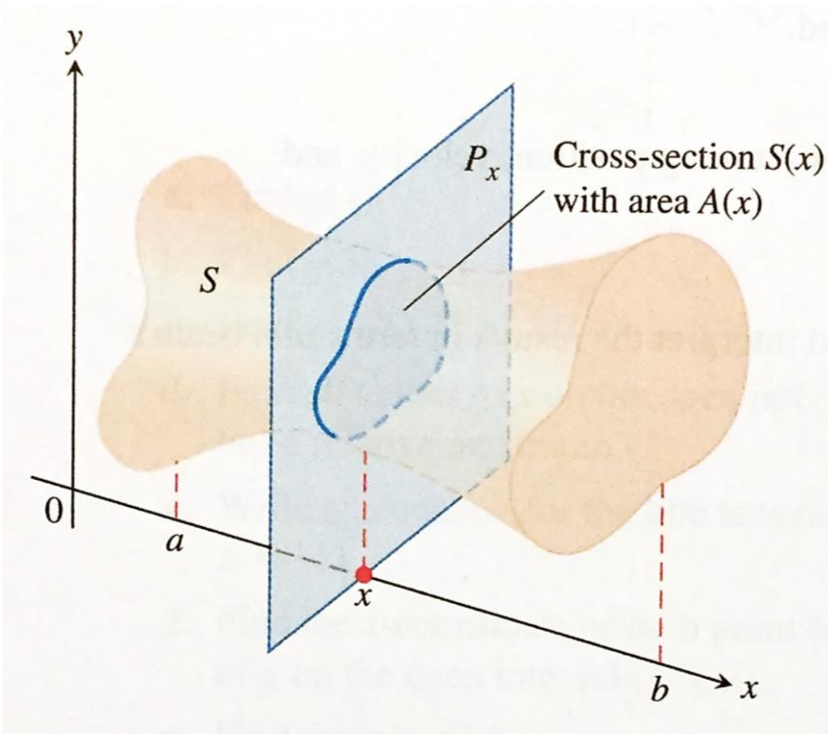
Computing volume using cross-sections
切断を用いて体積を測る

- Slicing by parallel planes (平行平面によりスライスをとる)

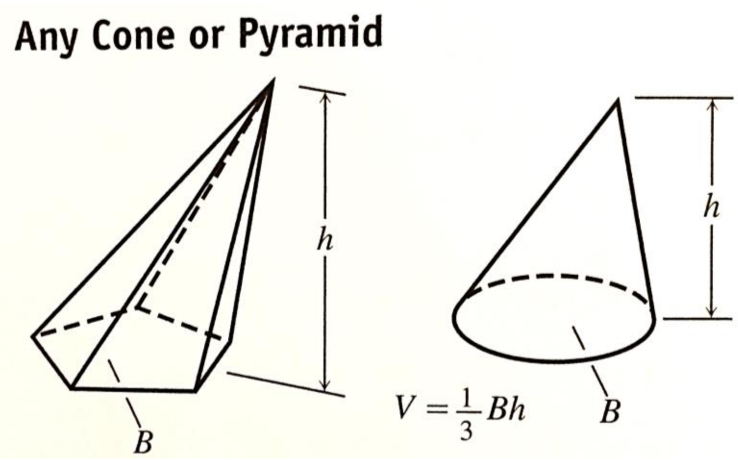


The volume of a solid of cross-section area $A(x)$ from $x = a$ to $x = b$ is:
(

$$\int_a^b A(x)\,dx = V$$

Parallel planes slicing: example I, cone

- Volume of cone of base B: (底面Bの錐の体積) :



$$A(x) = Area\ of\ slice\ at\ x$$

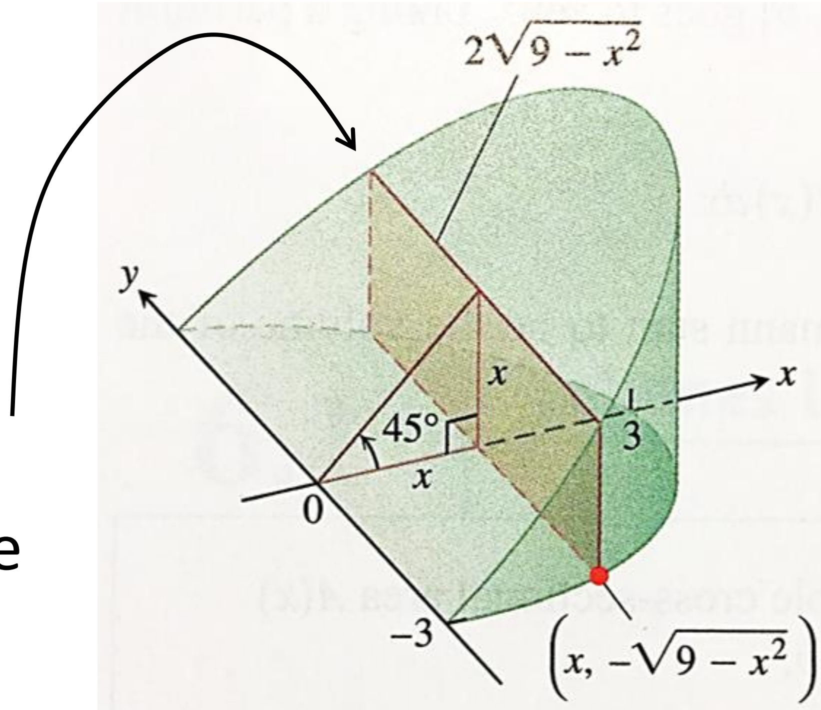
$A(x) = \pi \cdot x^2,$ (case of cylinder 円筒面)

$A(x) = 1/2 \cdot x^2 \cdot n \cdot \sin(2\pi/n)$ (case of n-正多角形)

Exercise: $Volume = V = \int \qquad dx =$

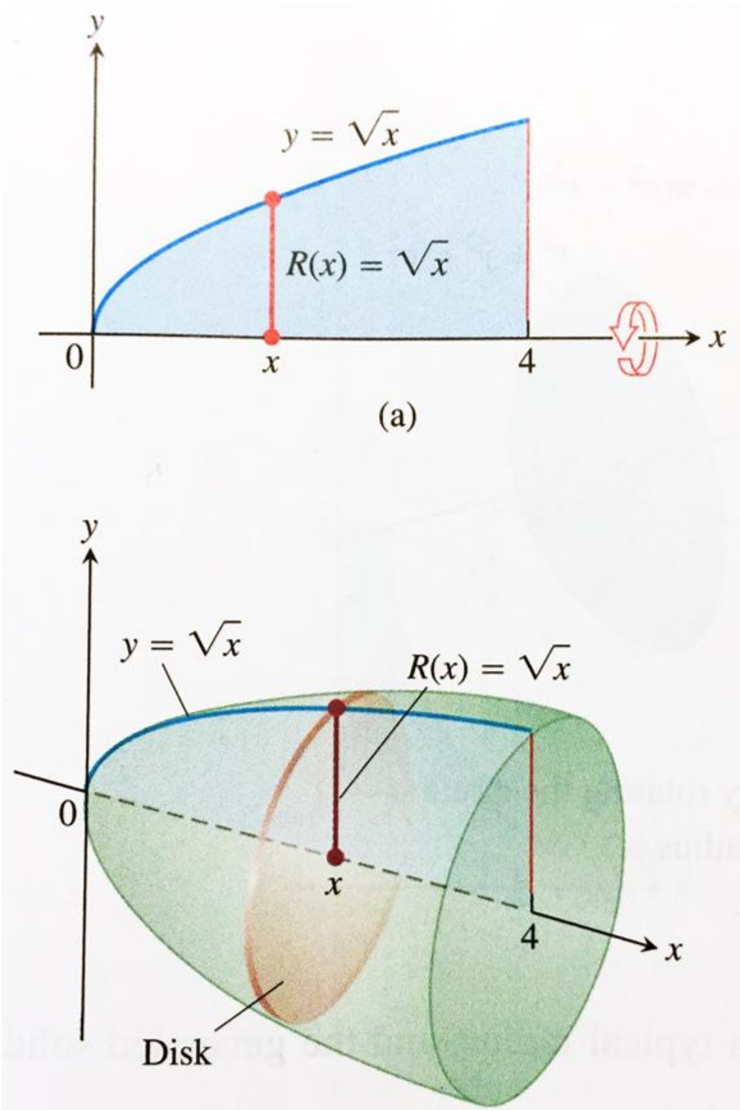
Parallel planes slicing: example 2

- We consider the part of cylinder (円筒） described in the image on the right:



- Slices are rectangles (長方形）
- What is the length and the width of the rectangle?
- Length $L(x) =$
- Width (幅） $W(x) =$
- Volume=$\int \quad L(x) \cdot W(x)\, dx$

Computing volume using cross-sections(II) 切断を用いて体積を計算する(II)



- Solids of revolution: the disk method
  (回転体：円板法)

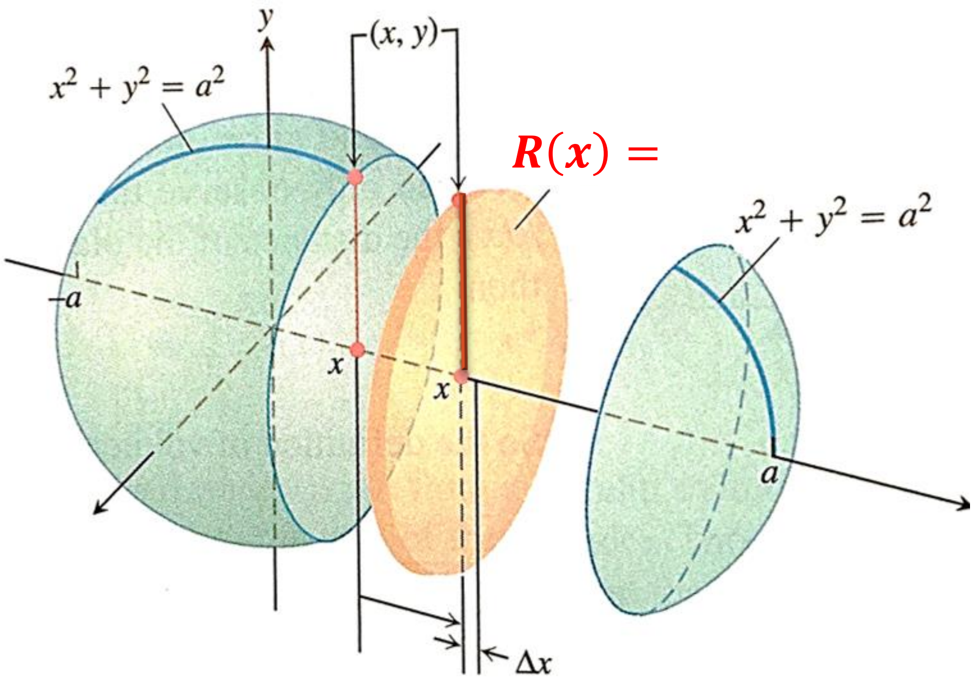
- The volume of a solid of revolution (回転体)between $x = a$ and $x = b$, whose disk at x has a radius (半径)of $R(x)$ is:

$$\int_a^b \pi R(x)^2 dx$$

- Example (left image):

$$\int_0^4 \pi R(x)^2 dx = \int_0^4 \pi x dx = \left[\frac{\pi x^2}{2}\right]_0^4 = 8\pi.$$

Solid of revolution: volume of the sphere



- What is the radius $R(x)$ of the disk in red of the sphere in the left?

$R(x) =$

- Deduce the volume using the formula for solids of revolution:
  $\int_{-a}^{a} \pi\, R(x)^2\, dx =$

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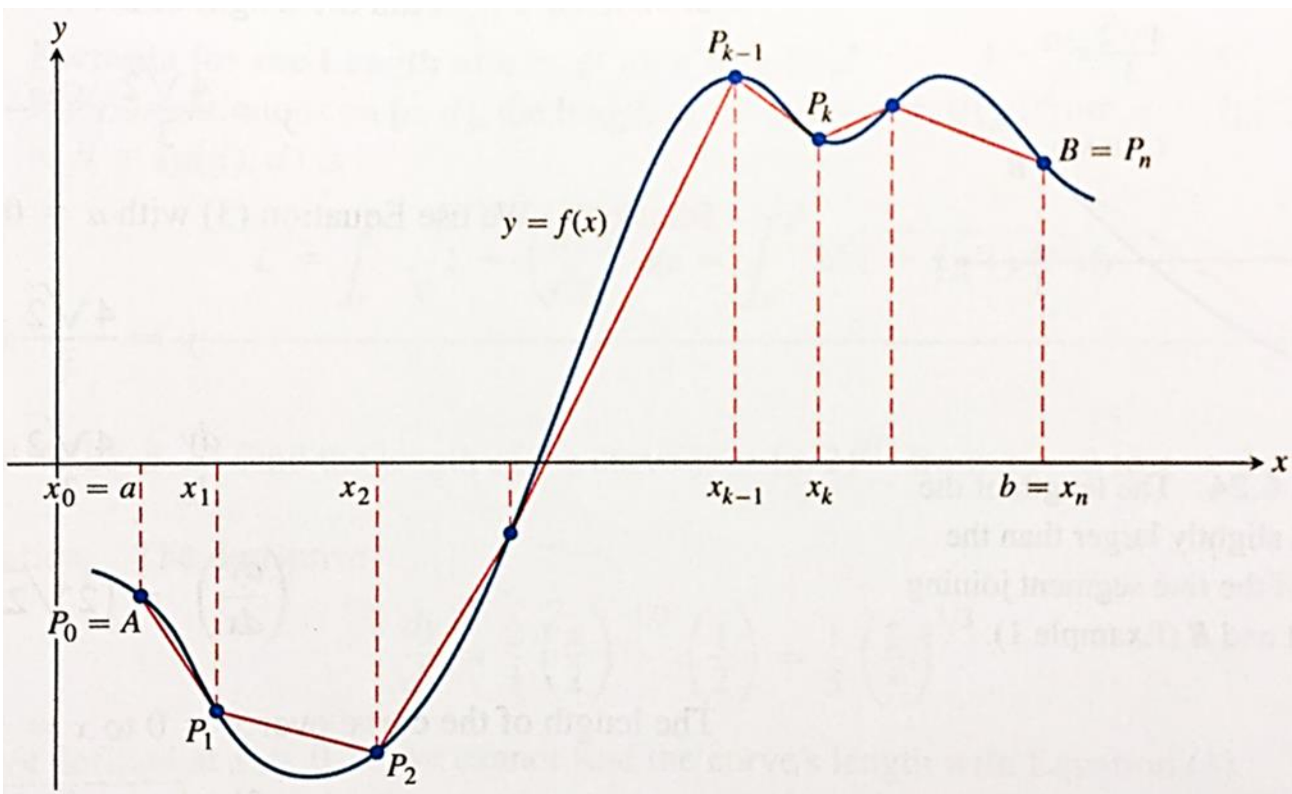
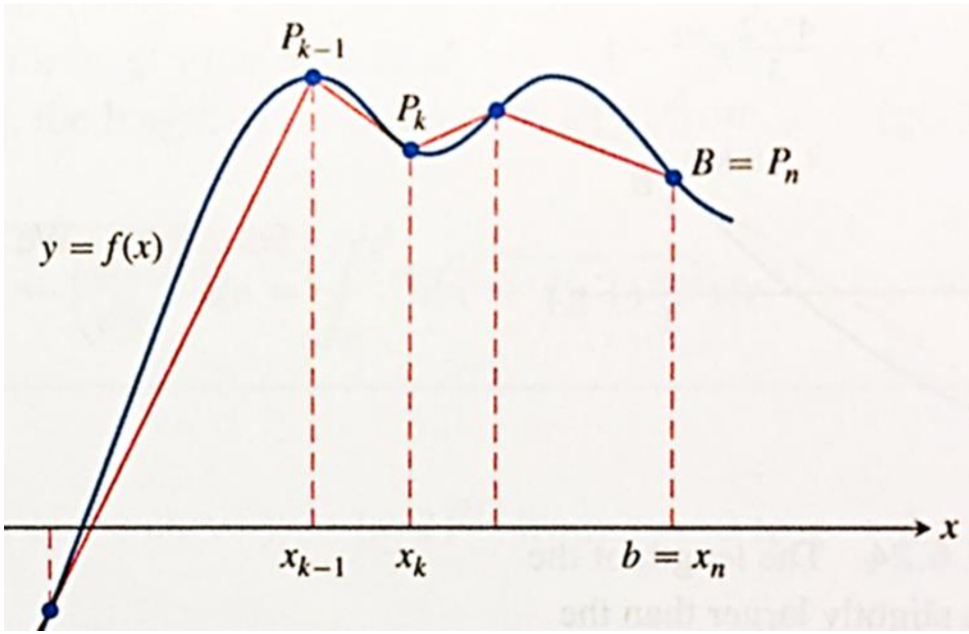
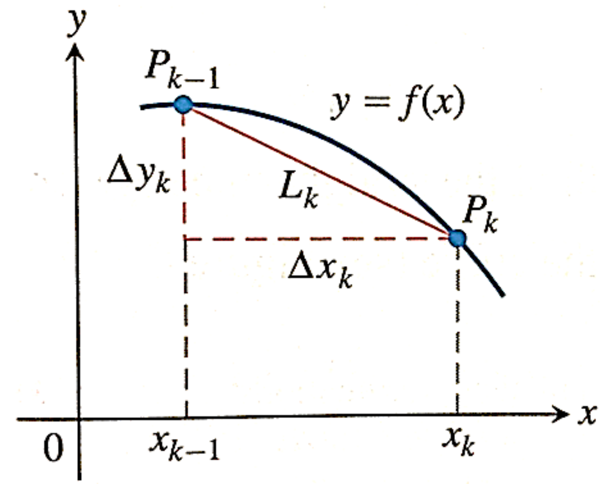
Application of integral: length of curves
積分の応用:曲線の長さ

- How to compute the length of a curve $y = f(x)$ between two points $x = a$ and $x = b$.
- <u>Assumption（仮定）: </u>f is differentiable (微分可能）



Length of curve



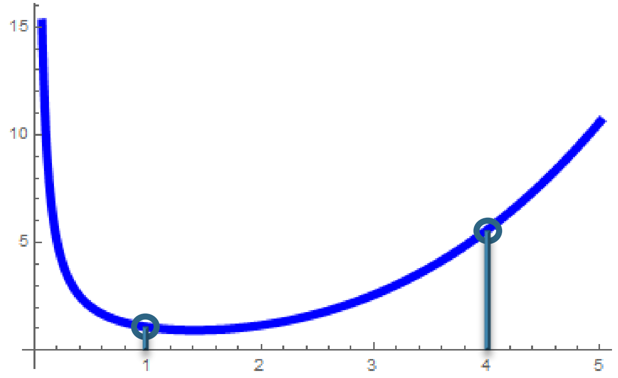


- $L_k = \sqrt{\Delta y_k^2 + \Delta x_k^2}$
- $Length \approx \sum_{k=1}^{n} L_k$

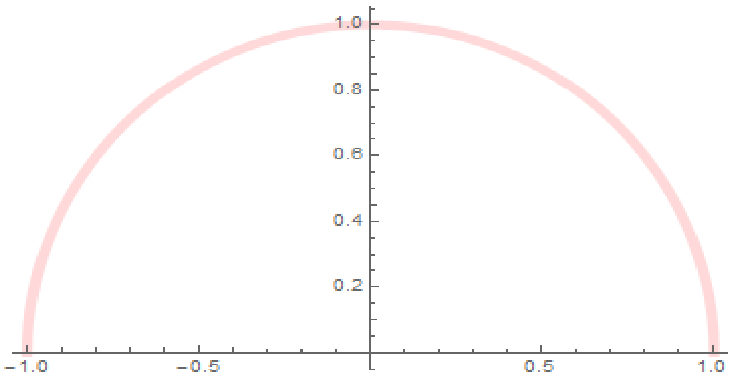
- $\Delta y_k = f(x_{k-1}) - f(x_k)$.
 Since f is differentiable, by the MVT (Mean Value Theorem, L4-2 page 6) there exists $x_{k-1} < c_k < x_k$ such that $\Delta y_k = f'(c_k)\Delta x_k$.
- $L_k = \Delta x_k \sqrt{f'(c_k)^2 + 1}$. And when $n \to \infty$, $\Delta x_k \to 0$:
- $Length = \lim_{n\to\infty} \sum_{k=1}^{n} \Delta x_k \sqrt{f'(c_k)^2 + 1} = \int_a^b \sqrt{f'(x)^2 + 1}dx$

Length of a curve: Exercise

- Compute the length of the curve between $x = 1$ and 4. $y = \frac{x^3}{12} + \frac{1}{x}$

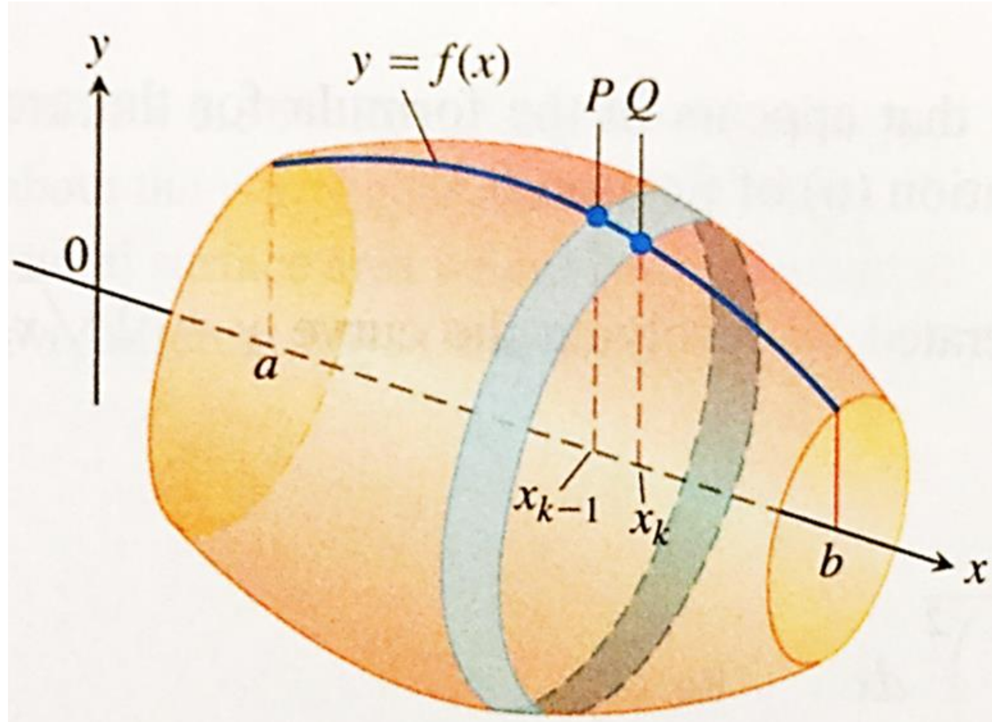


- Find the length of a half-circle: $y = \sqrt{1 - x^2}$.



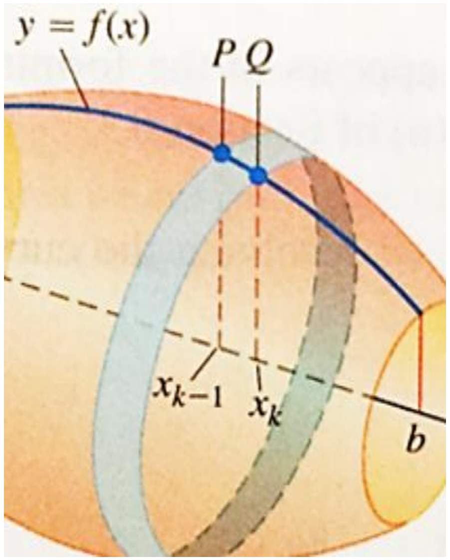
Areas of surface of revolution
回転面の面積

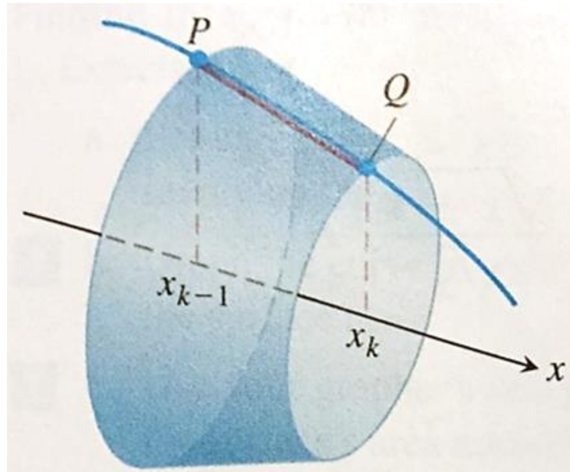
- Compute the area defined by the curve $y = f(x) > 0$ between $x = a$ and $x = b$, and revolved about the x −axis.



（$x = a$と$x = b$の間の曲線$y = f(x)$は、x軸を回転するからなる面積を測る）

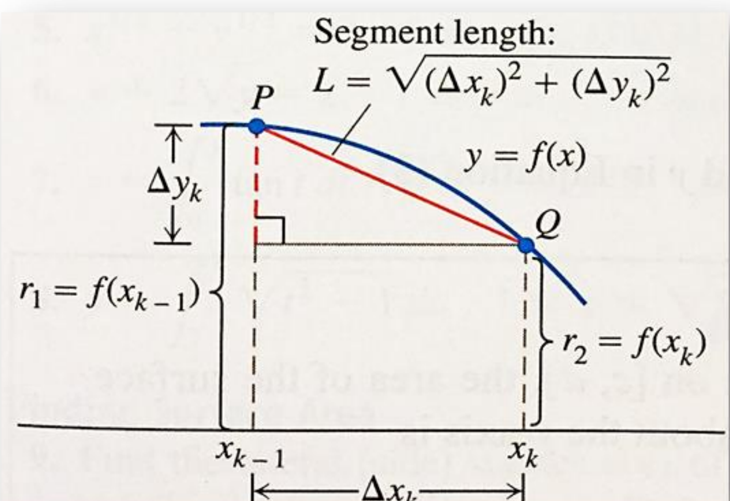
Areas of surfaces of revolution





- Area S_k of the *frustum* in blue（青い円錐台の面積）is equal to:

$$S_k = 2\pi\left(\frac{f(x_{k-1}) + f(x_k)}{2}\right)\sqrt{\Delta y_k^2 + \Delta x_k^2}$$



Making $n \to \infty$ gives

$$\lim_{n\to\infty}\sum_{k=1}^{n} S_k = \int_a^b 2\pi f(x)\sqrt{1 + f'(x)^2}\, dx$$

Area of a surface of revolution: Exercise

- Compute the area of a sphere of radius r.

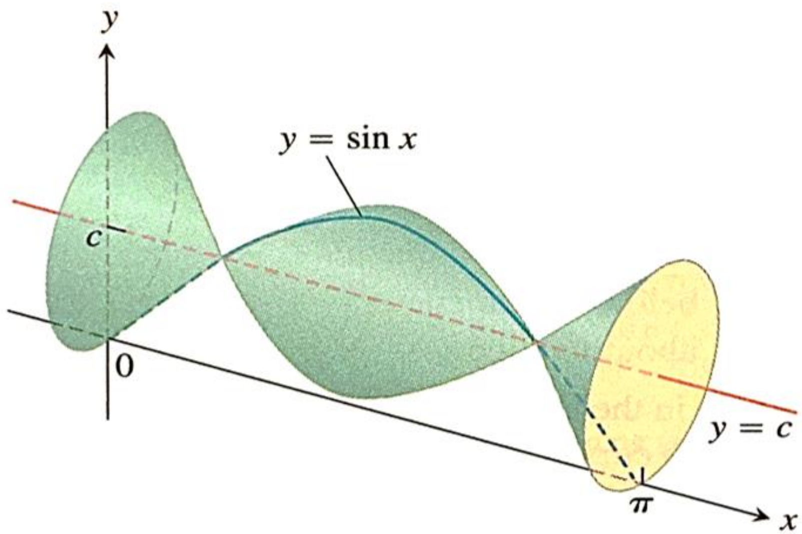
Homework: Hand in on June ??

1. Volume of revolution (回転体）

   The arch $y = \sin(x)$ is revolved about the line $y = c$ (for $0 \leq c \leq 1$) to form a solid as shown in the figure (曲線$y = \sin(x)$の弧は$y = c(0 \leq c \leq 1)$の軸を回転することによって得られる回転体）

   a. Find the value of c that minimizes the volume. What is the minimum volume ?

   b. What value of $c \in [0,1]$ maximizes the volume ?



Homework: Hand in on June ??

2. The shaded band shown here is cut from a sphere of radius R by parallel planes h units distant apart.
   以下の陰のついた輪がhユニット離れている平行な平面に半径R球面から切れる。
   Show that the surface area of the band is $2\pi Rh$
   輪の面積が$2\pi Rh$であることを示せ。