Essential Mathematics for Global Leaders I

Lecture 5-1

Integration I

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Plan (tentative)

[4/13] L1 : introduction. Review of high-school mathematics in English.

[4/20-27] L2-3 : Functions and graphs. Plotting with Mathematica I (グラフをプロットする)

[5/7] L4 : Infinitely small and large : limits (極限)

[5/11] L5 : Differentiation (連続性と微分法)

[5/18] L6 : Differentiation II : extrema, related rates ...（極値と...）

[5/25] L7 : Differentiation III : Newton’s method, Taylor’s expansion (ニュートン法とテイラー展開）

[6/1] L8 : Mid-term test. Integration I : definition, fundamental theorem of calculus　積分I.

[6/8] L9 : computation of indefinite integrals　不定積分

[6/15] L10 : Application of Integration I : length, volume and surfaces
積分の応用:長さ、面積、体積

[6/22] L11 : Application of Integration II : average, center of mass　（質量中心）, work of a force.

[6/29] L12 : Ordinary Differential equations (one variable)　常微分方程式

[7/6] L13 : Linear Differential Equations of order 2 : harmonic oscillators (small-angle pendulum, spring). 二階線形微分方程式:調和振動子　(振幅が小さい振り子、ばね)

[7/13] L14 : Ordinary Differential Equations with Mathematica. Mathematicaを利用して常微分

Content Integration I

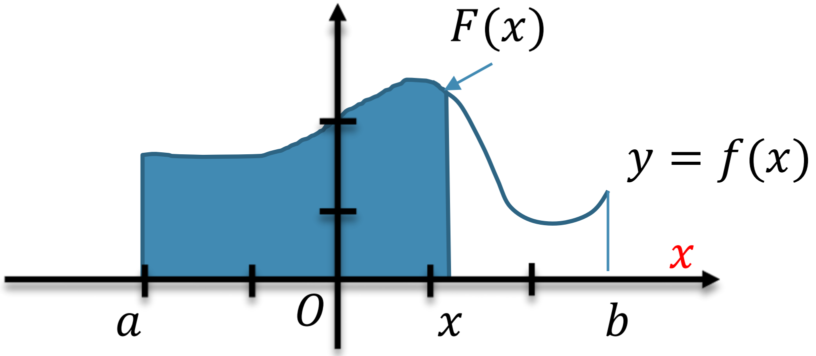
1. Area under a curve, definition, examples
（グラフとx −軸の間の面積、定義、例）。

2. Fundamental Theorem of Calculus
（微分積分学の基本定理）

3. Calculation of Primitives 1
(原始関数の計算）

Introduction: Area under the graph of a function (関数のグラフとx-軸の間の面積)

Goal: Let $f\colon [a,b] \to \mathbb{R}$, $f > 0$ on $[a,b]$.
Define a function a function $F\colon [a,b] \to \mathbb{R}$ that measures the surface as shown below
(以下の図のような面積を測る関数$F\colon [a,b] \to \mathbb{R}$を定義したい)。



Notation (記号): $\boxed{F(x) = \int_a^x f(t)dt}$

Fundamental Theorem of Calculus (微分積分学の基本定理) :

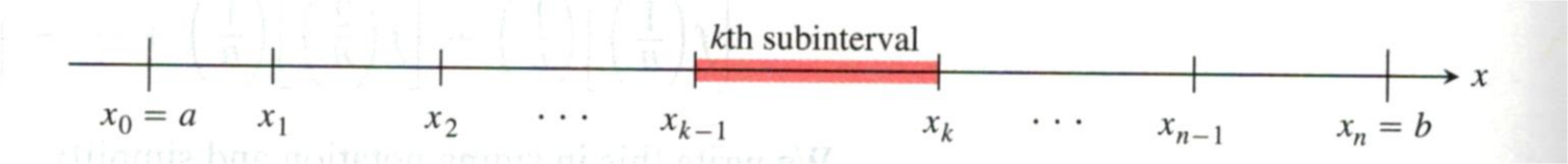
$$\boxed{\boldsymbol{F'(x) = f(x)}}$$

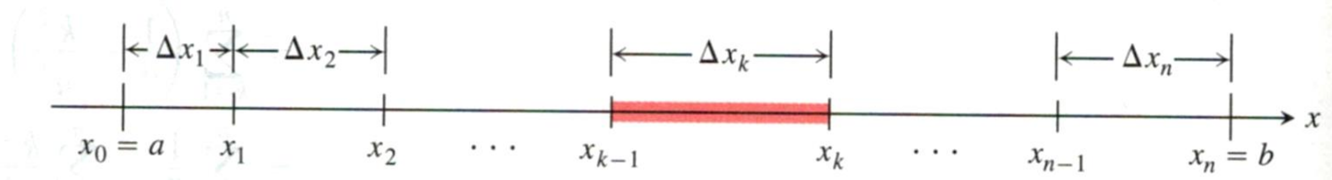
Partition of an interval (**区間の分割**）

- **Definition:** A **partition** P of the interval $[a, b]$ is a sequence of numbers $P = \{x_0, \dots, x_n\}$ like this:

  $$a = x_0 < x_1 < x_2 < \cdots < x_n = b$$

  The **mesh** (**メッシュ**） of $P$is: $\delta(P) = \max(x_{i+1} - x_i)$ for $0 \le i < n$



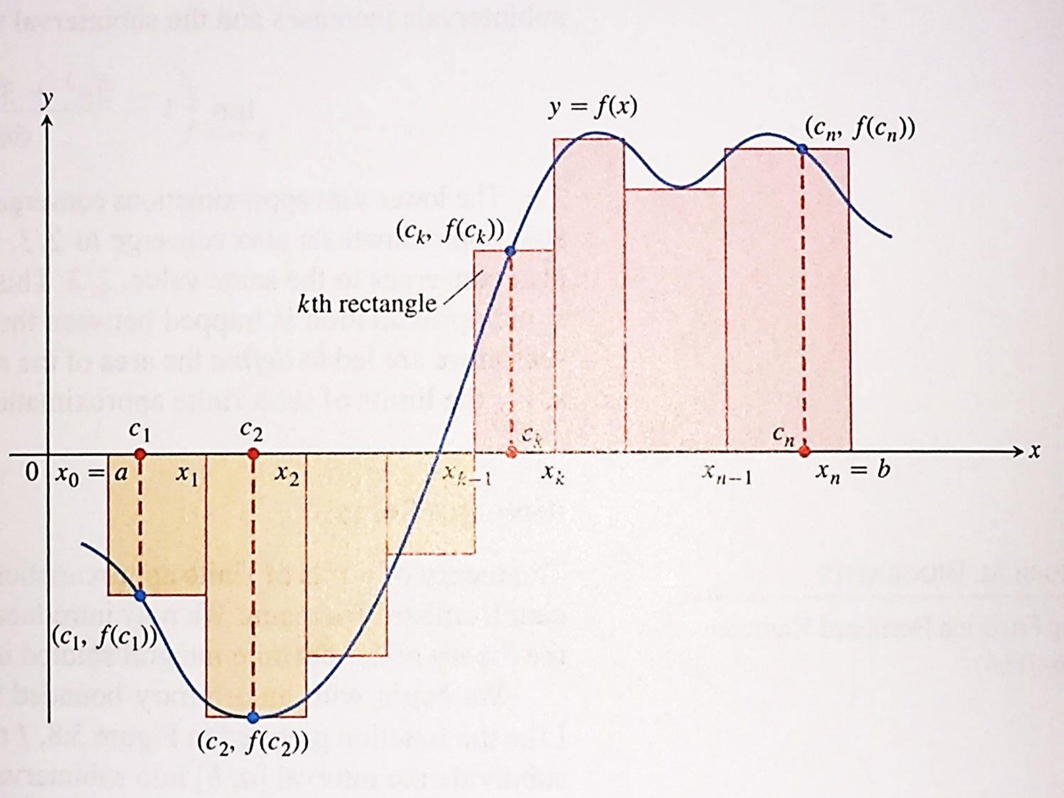


Mesh

$= \max_{k} \Delta x_k$

Riemann’s sum (リーマン和）

- A partition P allows to approximate the surface using Riemann’s sums:
  In each subinterval $[x_{k-1}, x_k]$ choose a point c_k

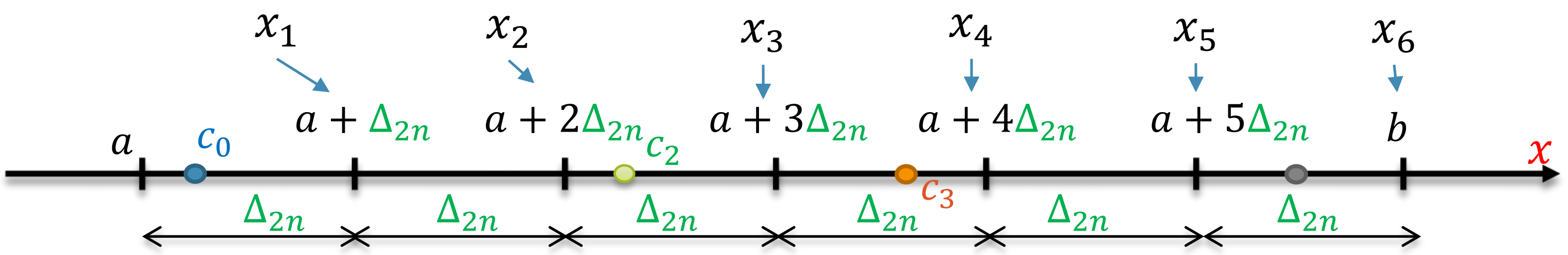


- The area in the k-th rectangle is
  $f(c_k)\Delta_k$
  $= f(c_k)(x_{k+1} - x_k)$
- Riemann’s sum

$$S(P) = \sum_{k=1}^{n} f(c_k)\Delta_\mathrm{k}$$

Partition of equal length. Infinite sum

- Let $\Delta_n = \frac{b-a}{n}$. Consider the sequence of decreasing partitions $(P_n)_{n\in\mathbb{N}}$,
  $P_{2n} = \{x_0 = a, a + \Delta_n, a + 2\Delta_n\,, \cdots, a + (n-1)\Delta_n, b = x_n\}$



- For any choice of points $c_k \in [x_k, x_{k+1}]$, if the Riemann's sum

$$S(P_n) = \sum_{k=0}^{2n-1} f(c_k)\Delta_n = \frac{b-a}{n}\sum_{k=0}^{n-1} f(c_k)$$

Integrable functions (積分可能な関数）

- **Assumption (仮定）:**
  f is continuous (連続）on $[a, b]$
  (for any $c \in [a, b]$, $\lim_{x \to c} f(x) = f(c)$)

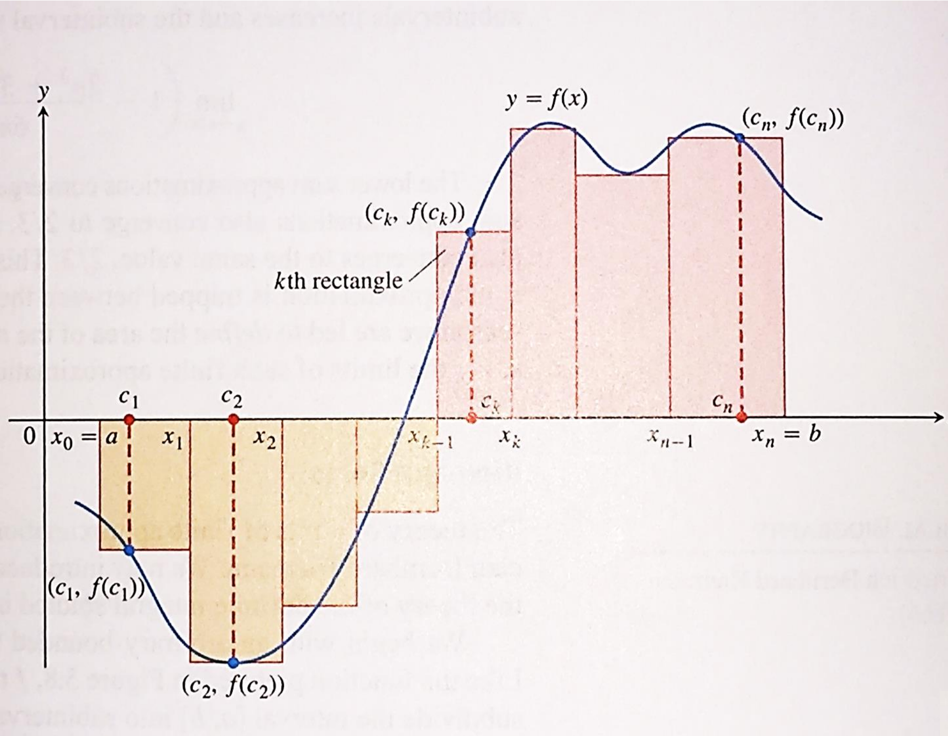
- If $\lim_{n \to \infty} S(P_n) = \lim_{n \to \infty} \sum_{k=0}^{n-1} f(c_k) \Delta_n = J$, then:

$$\boxed{J = \int_a^b f(t) dt}$$

  is called the integral of f between a and b.

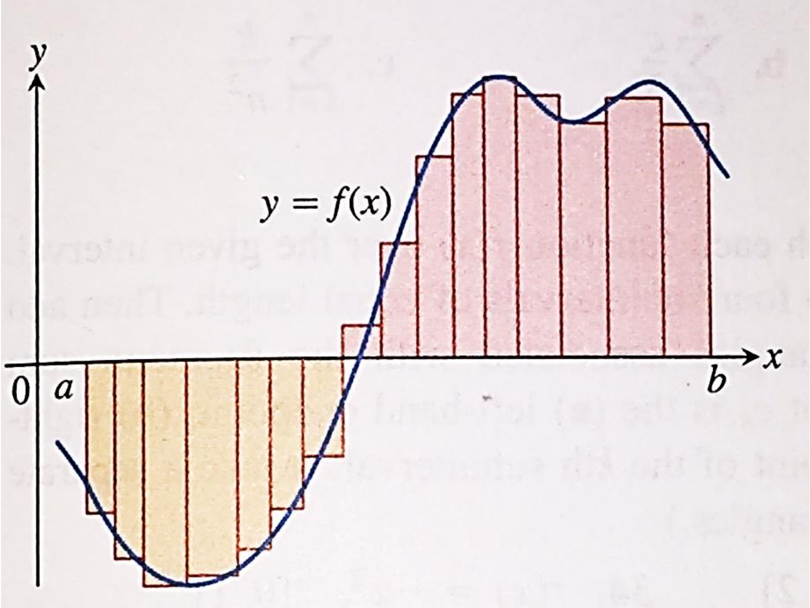
- **Remark:** $S(P_n)$ depends on the points c_0, c_1, c_2 ... but the value J does not.
  $S(P_n)$ は点$c_0, c_1, \dots$に依存するがJは依存しない.

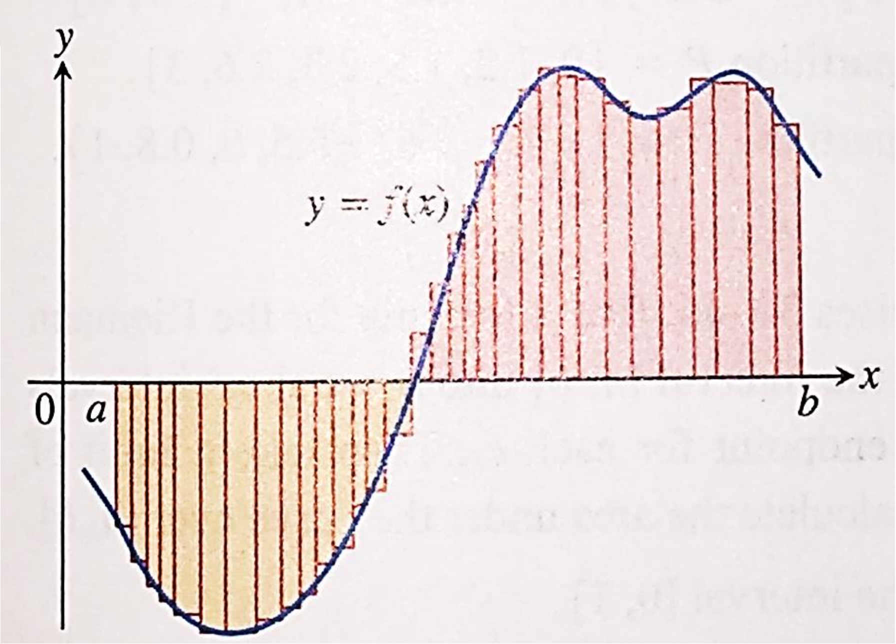
- **Theorem:** Contiuous functions on $[a, b]$ are integrable.



More generally
J does not depend on the sequence of partitions P_n chosen to take the limit.

より一般的に、Jは極限をとるたまの分割の列P_nに依存しない。





Rules satisfied by integrals
積分が満たす法則

Zero Width Interval (幅0区間)	$\int_a^a f(x)dx = 0$	**Fig 1**
Additivity（加法性）	$\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)\,dx$	**Fig 2**
Max-Min equality (最大･小の等式)	Assume that f has a min m and a max M on $[a, b]$ $m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$	**Fig 3**

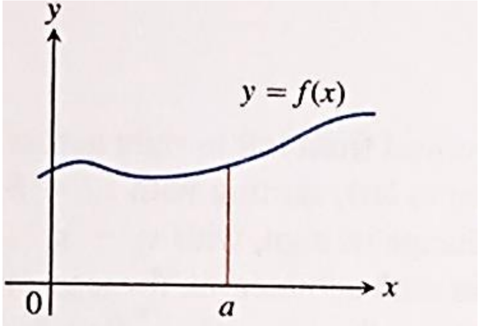


Fig. 1

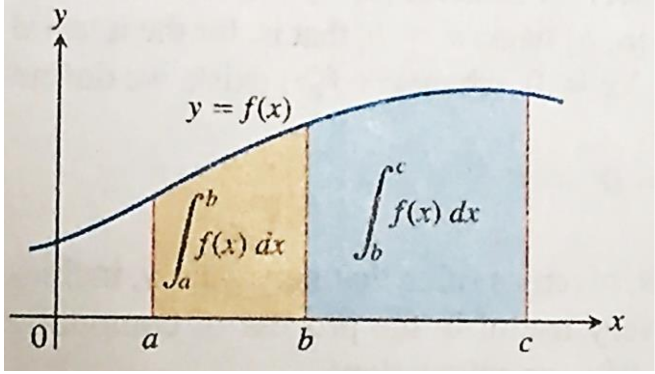


Fig. 2

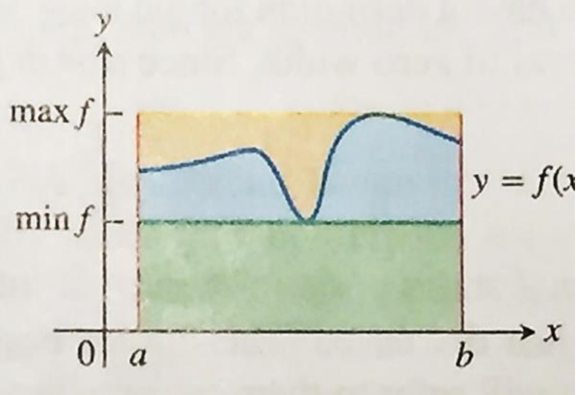


Fig. 3

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2. Fundamental Theorem of Calculus （微分積分学の基本定理）
3. Calculation of Primitives 1 (原始関数の計算）

Fundamental theorem of Calculus (FTC) (微分積分学の基本定理)

- **(Mean Value Theorem for integral 積分の平均値定理):**

Let $f\colon [a,b] \rightarrow \mathbb{R}$ be a continuous function. There exists a point $c \in [a,b]$ such that:

$$f(c) = \frac{1}{b-a}\int_a^b f(x)dx$$

- **FTC**: Given a continuous function $f\colon [a,b] \rightarrow \mathbb{R}$, $F(x) = \int_a^x f(t)dt$ is differentiable on $[a,b]$ and:

$$F'(x) = f(x)$$

- The function $F(x) = \int_a^x f(t)\,dt$ is a primitive (原始関数） or an antiderivative (不定積分）

Exercise

- Use the FTC to compute the derivative $F'(x)$ of F:

$$F(x) = \int_1^{x^2} \cos t \ dt$$

Consequence of the FTC

Function	**(indefinite) integral**
$\int x^n dx$	$\frac{x^{n+1}}{n+1} + C$
$\int \frac{dx}{x}$	$\ln\lvert x\rvert + C$
$\int e^x\, dx$	$e^x + C$
$\int a^x dx$	$\frac{a^x}{\ln(a)} + C$
$\int \sin x\;\; dx$	$-\cos x + C$
$\int \cos x\, dx$	$\sin x + C$

Definite & Indefinite integral (定積分 & 不定積分）

- Given a function $f\colon [a, b] \to \mathbb{R}$, a function $F\colon [a, b] \to \mathbb{R}$ such that $F'(x) = f(x)$ is called an antiderivative (不定積分） or primitive function （原始関数）.

- If F_1 and F_2 are two primitive functions of f, then

$$F_1(x) = F_2(x) + C, \qquad C \in \mathbb{R}$$

(Consequence of Corollary 2 of Lesson 4 - 2, page 7)

- A common notation （普通の記号） for primitives is:

$$\int f(t)dt = F(x) + C, \qquad C \in \mathbb{R}$$

$\int f(t)dt$ is called "indefinite integral".

FTC for definite integrals

- **FTC II:** Let $f\colon [a,b] \to \mathbb{R}$ be a continuous function and F a primitive function of f. Then
$$\int_a^b f(t)dt = F(b) - F(a) \quad \left(= [F(x)]_a^b\right)$$

- **Exercise:** Evaluate the following integrals:

1. $\int_1^4 \left(\frac{3}{2}\sqrt{x} - \frac{4}{x^2}\right) dx$

2. $\int_1^{\pi/4} (\tan x)^2 \, dx$

Proof of the FTC 微分積分基本定理の証明

f continuous on $[a,b]$, $F(x)=\int_a^x f(t)\,dt \Rightarrow F'(x)=f(x)$.

- $F(x+h)-F(x)=\int_a^{x+h} f(t)dt-\int_a^x f(t)\,dt$
$$=\int_x^{x+h} f(t)dt$$
- By the MVT for integral, there exists $c\in[x,x+h]$ $\int_x^{x+h} f(t)dt=h\cdot f(c)$. Thus $\frac{F(x+h)-F(x)}{h}=f(c)$.
- Since f is continuous $\lim_{h\to 0} f(c)=\lim_{c\to x} f(c)=f(x)$, and $F'(x)=\lim_{h\to 0}\frac{F(x+h)-F(x)}{h}=f(x)$.

Mathematica command: **Integrate**

To compute indefinite integrals (antiderivation)

Ex: $\ln \int \ln(x)\, dx$

```
In[1]:= Integrate[Log[x],x]
Out[1]:= -x+x Log[x]
```

To compute definite integrals (between an interval)

Ex: $\int_{0}^{\frac{\pi}{2}} \sin(x)\, dx$

```
In[2]:= Integrate[Sin[x],{x,0,Pi/2}]
Out[2]:= 1
```

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The chain rule and substitution rule （連鎖律と置換律）

- **Substitution rule** (= Chain rule backward)
- If $u = g(x)$ is a differentiable function whose image (像） is an interval I, and if f is continuous on I, then:

$$\int f(g(x))g'(x)dx = \int f(u)du$$

- **Example:** $\int (x^3 + x)^5(3x^2 + 1)dx.$

Let $u = x^3 + x$. Then, $du = \frac{du}{dx}dx = (3x^2 + 1)dx$,

and $\int (x^3 + x)^5(3x^2 + 1)dx = \int u^5 du.$

This gives $\frac{u^6}{6} + C = \frac{(x^3+x)^6}{6} + C$

Examples of applications of the substitution rule

Function to integrate	Indefinite integral	Example of function $u(\mathrm{x})$
$\int u' u^n dx$	$\frac{u^{n+1}}{n+1} + C$	$u = \cos(x)$ $\int \cos x \, (\sin x)^n dx = -\frac{(\sin x)^{n+1}}{n+1} + C$
$\int \frac{u'}{u} dx$	$\ln\lvert u\rvert + C$	$u = \cos x, \int \frac{\sin x}{\cos x} dx = -\ln\lvert\cos x\rvert + C$
$\int u' e^u \, dx$	$e^u + C$	$u = 1/x \,, \int \frac{e^{\frac{1}{x}}}{x^2} dx = -e^{\frac{1}{x}} + C$
$\int u' \sin u \;\, dx$	$-\cos u + C$	$u = x^2, \int x \sin x^2 \;\, dx = \cos x^2 + C$

Exercise (Substitution rule)

1. $\int \sqrt{2x+1}dx$

2. $\int \cos(7\theta+3)d\theta$

3. $\int x\sqrt{2x+1}dx$

Substitution of definite integral (定積分の置換律)

- **Theorem**: If g' is continuous on the interval $[a, b]$ and f is continuous on the image (像) of $g(x) = u$, then:

$$\int_a^b f\big(g(x)\big) \cdot g'(x)dx = \int_{g(a)}^{g(b)} f(u)du$$

- **Example:** $\int_{-1}^{1} 3x^2\sqrt{x^3+1}\,dx$

$$u = x^3 + 1 \Rightarrow du = \frac{du}{dx}dx = (3x^2)dx$$

$\Rightarrow$ when $x = -1, u = (-1)^3 + 1 = 0$

$\Rightarrow$ and when $x = 1, u = 2.$

$$\int_{-1}^{1} 3x^2\sqrt{x^3+1}\,dx = \int_0^2 \sqrt{u}\,du = \left[\frac{2}{3}u^{\frac{3}{2}}\right]_0^2 = \frac{2}{3} \cdot 2\sqrt{2}$$

Exercise

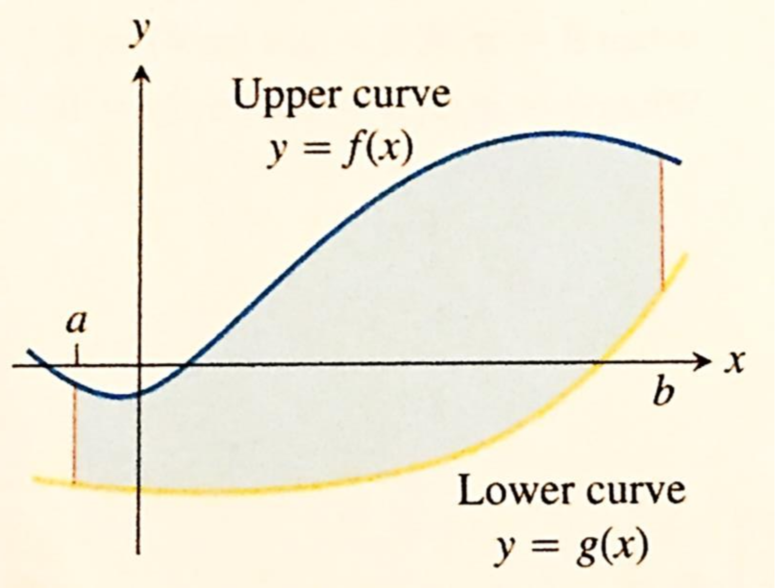
- Evaluate the definite integral

$$\int_0^{\pi} 3(\cos(x))^2 \sin(x)\,dx$$

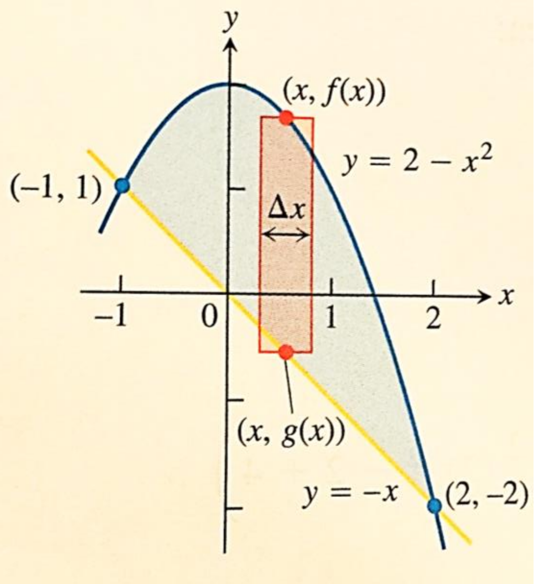
Area between two curves (曲線の間の面積）

- **Definition**: If f and g are continuous functions with $f(x) \geq g(x)$ then the area of the region between the curves $y = f(x)$ and $y = g(x)$ from a to b is:

$$\int_a^b [f(x) - g(x)]dx$$



- **Exercise:** Find the area of the region in blue closed by the parabola$y = 2 - x^2$ and $y = -x$.



Integration by parts (部分積分)

- By integrating $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$, we obtain
  $$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx$$

- If we let $u = f(x)$, and $v = g(x)$, then $du = \frac{du}{dx}dx = f'(x)dx$ and $dv = \frac{dv}{dx}dx = g'(x)dx$.
  $$\int udv = uv - \int vdu$$

- **Example:** $\int x\cos(x)\,dx$
  Let $u = x,$ and $dv = \cos(x)\,dx$.
  Then $du = dx$ and $v = \sin(x)$.
  $$\int x\cos(x)\,dx = x\sin(x) - \int \sin(x)\,dx$$
  $$= x\sin(x) - \cos(x) + C$$

Exercise about integration by parts

- The difficulty is to find the functions u and dv.

1. $\int \ln(x)\, dx$

2. $\int x^2 e^x dx$ (Hint: apply twice integration by parts)

3. $\int x \ln(x)\, dx$