Essential Mathematics for Global Leaders I

Lecture 4-3

Differentiation III

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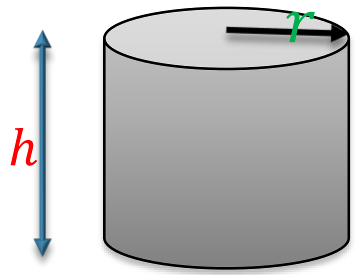
Solution to exercise Lecture 4-2 page 11



Given a can（缶, = a cylinder 円筒）

Constraint: Volume=$h\pi r^2$=1000cm³

Find h and r that minimizes the surface (面積= $S(r,h)$)of the can:

$$\min_{h\pi r^2=1000} S(r,h)$$

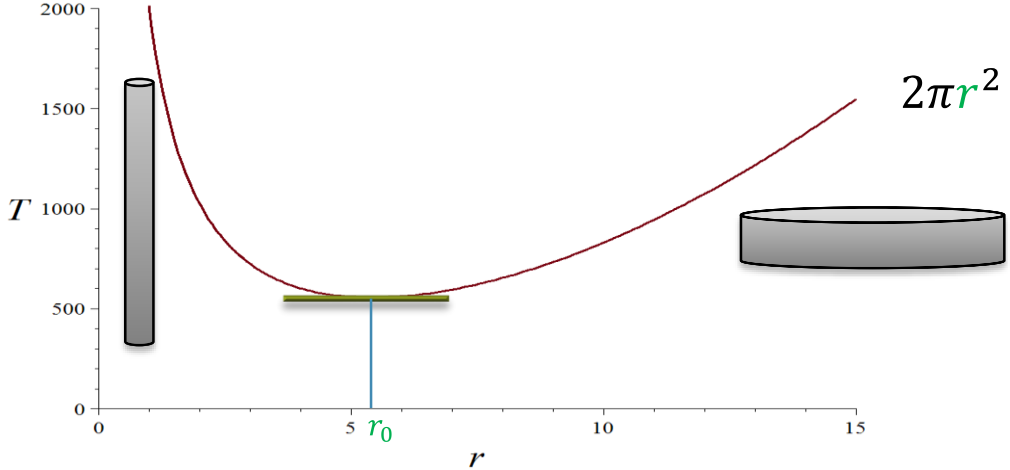
Answer: $r = \sqrt[3]{\frac{500}{\pi}} \approx 5.42cm \qquad h = 2r \approx 10.84cm$

$$S(r,h) = 2\pi r^2 + 2\pi rh$$
$$h\pi r^2 = 1000 \Rightarrow h = 1000/\pi r^2$$
$$S(r,h) = 2\pi r^2 + 2000/r = T(r)$$

Find the minimal value of T (関数Tの最低値をとろう): page 27. Find r_0 such that $T'(r_0) = 0$ (r_0 is a critical point, 臨界点）:

$T'(r) = 4\pi r - 2000/r^2$ therefore $T'(r) = 0$ is equivalent to $4\pi r_0 = \frac{2000}{r_0^2}$.

Solution to exercise Lecture 4-2 page 11



$$2\pi r^2 + 2000/r = T(r)$$

We find $r_0^3 = 500/\pi$ and thus $r_0 = \sqrt[3]{500/\pi}$ and $h = \frac{1000}{\pi(500/\pi)^{2/3}}$

$$h = 2 \cdot (500)^{1/3}/\pi^{1/3} = 2\sqrt[3]{500/\pi} = 2r_0$$

注意: $T'(r_0) = 0$ means that maybe r_0 is a
- maximum,
- minimum, or
- neither of them.

To decide, let us compute the 2nd derivative:
$T''(r) = 4\pi + 1000/r^3 > 0 \text{ on } \mathbb{R}_+$.

Thus, T' is increasing and the critical point r_0 is a local minimum

Plan (tentative)

[4/13] L1 : introduction. Review of high-school mathematics in English.

[4/20-27] L2-3 : Functions and graphs. Plotting with Mathematica I (グラフをプロットする)

[5/7] L4 : Infinitely small and large : limits (極限)

[5/11] L5 : Differentiation (連続性と微分法)

[5/18] L6 : Differentiation II : extrema, related rates ...（極値と...）

[5/25] L7 : Differentiation III : Newton's method, Taylor's expansion (ニュートン法とテイラー展開）

[6/1] L8 : Mid-term test. Integration I : definition, fundamental theorem of calculus　積分I.

[6/8] L9 : computation of indefinite integrals　不定積分

[6/15] L10 : Application of Integration I : length, volume and surfaces
積分の応用：長さ、面積、体積

[6/22] L11 : Application of Integration II : average, center of mass （質量中心）, work of a force.

[6/29] L12 : Ordinary Differential equations (one variable)　常微分方程式

[7/6] L13 : Linear Differential Equations of order 2 : harmonic oscillators (small-angle pendulum, spring). 二階線形微分方程式：調和振動子（振幅が小さい振り子、ばね)

[7/13] L14 : Ordinary Differential Equations with Mathematica. Mathematicaを利用して常微分

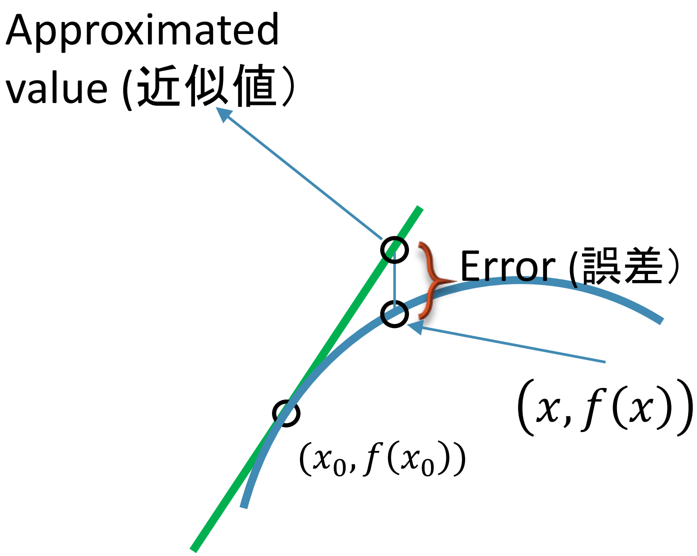
Program L4-3

1. Approximating functions by polynomials:
   Taylor's expansion
   多項式を用いて関数を近似する:テイラー展開

2. Approximating zeroes of functions:
   Newton's method
   関数の零を近似する:ニュートン法

1^{st} order (or linear) approximation 一次近似　or 線型近似

- Idea (概念): If I know $f(x_0)$ and $f'(x_0)$.
  I can approximate $f(x)$ when x is near to x_0 by:
  $$f(x) \approx f(x_0) + (x - x_0)f'(x_0)$$



- How good/bad is it?
  Given x_0, x fixed, let $c \in \mathbb{R}$ such that
  $$f(x) = f(x_0) + f'(x_0)(x - x_0) + c(x - x_0)^2$$

c depends on x and x_0 (cは変数xもx_0に依存する)

Problem: what is c ?

Estimation of the error（誤差の推定）

- $f(x) = f(x_0) + f'(x)(x - x_0) + \underbrace{c(x - x_0)^2}_{\text{Error term}} \qquad c \in (x_0, x)$

To estimate c let us fix x and replace x_0 by a variable y .
(cを推定するために、xを不動とし、x_0を変数yに換える）

$$R_x(y) := f(x) - (f(y) + (x - y)f'(y) + c(x - y)^2)$$
$$R_x(x) = 0 = R_x(x_0)$$
$$R'_x(y) = -f''(y)(x - y) + 2c(x - y)$$

- By the mean value theorem (Lect. 4-2 page 6) there exists $d \in (x_0, x)$ such that: $R_x(x) - R_x(x_0) = 0 = R'_x(d)(x - x_0)$
$$\Rightarrow R'_x(d) = 0 = -f''(d)(x - d) + 2c(x - d)$$
- $\Rightarrow \boxed{c = \dfrac{f''(d)}{2}}$, $\quad d \in (x_0 , x)$
- The error of approximation $\frac{f''(d)}{2}(x_0 - x)^2$ verifies: $\lim_{x \to x_0} \frac{R_x(x_0)}{x - x_0} = 0$

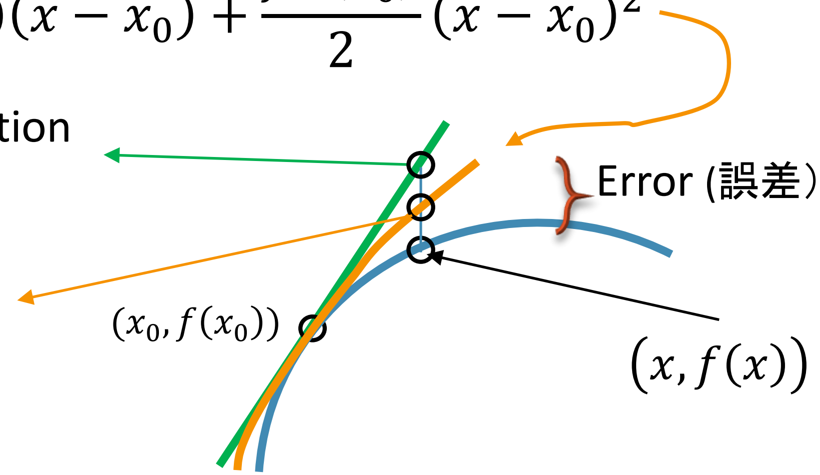
Second order approximation
二次近似

- For x close to x_0:

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2$$

1st order approximation
(一次近似)

2nd order approximation
(二次近似)



- Error of approximation:

$$R_x(y) := f(x) - f(y) - f'(y)(x - y) + f''(y)(x - y)^2 - c(x - y)^3.$$

c is chosen so that $R_x(x_0) = 0$.

Same computations as in the previous page give:

$$c = \frac{1}{6} f'''(d), \qquad d \in (x_0, x)$$

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2 + \frac{f'''(d)}{6}(x - x_0)^3$$

Higher order approximation: Taylor's expansion (Taylor展開)

- **Definition**: Taylor expansion of f at order n around x_0

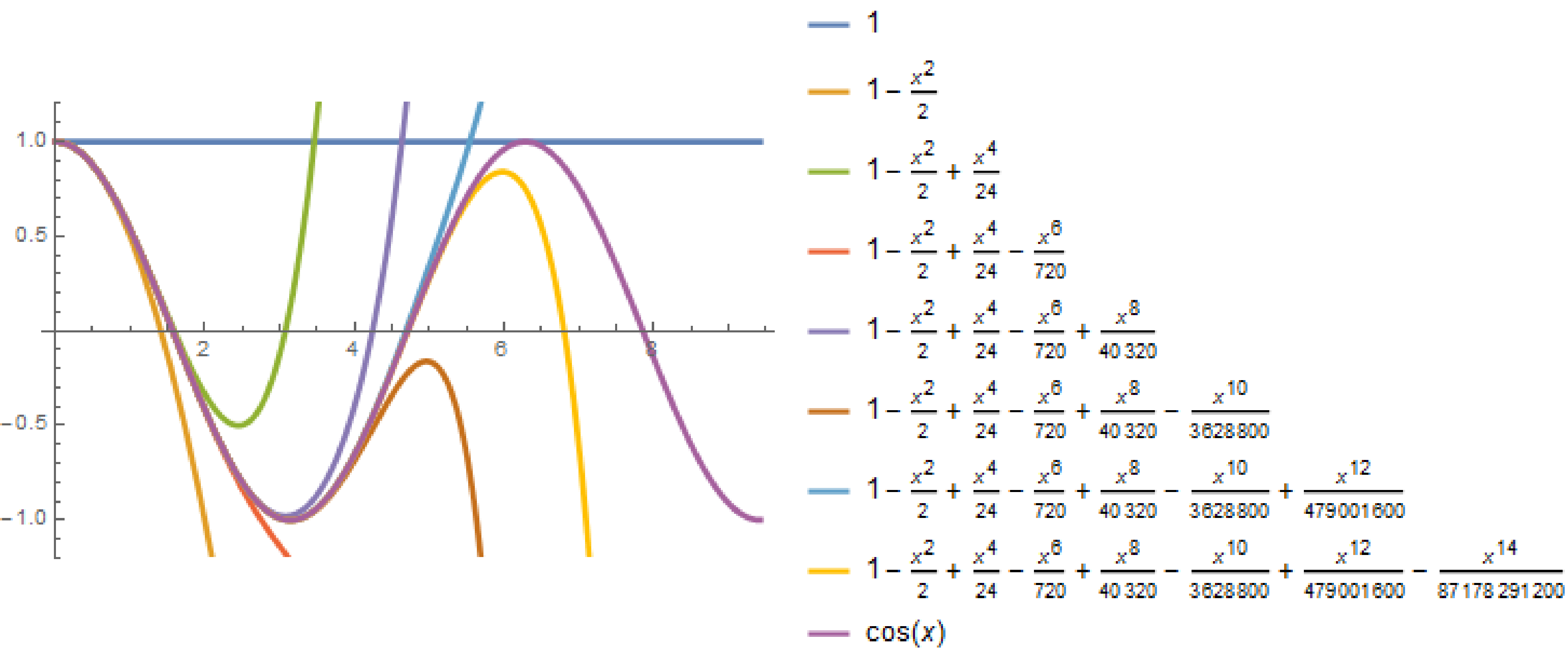
$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2}(x - x_0) + \cdots + \frac{1}{(n+1)!} f^{(n+1)}(d)(x - x_0)^{n+1}$$

- **Definition:** $P_n(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2}(x - x_0) + \cdots + \frac{(x-x_0)^n}{n!} f^{(n)}(x_0)$

is polynomial of degree n called Taylor polynomial.

$\epsilon_n(x) = \frac{1}{(n+1)!} f^{(n+1)}(d)(x - x_0)^{n+1}$ is called the Taylor-Lagrange remainder (剰余）。

Cosine and its Taylor expansions at order 1 to 15 at $x = 0$.

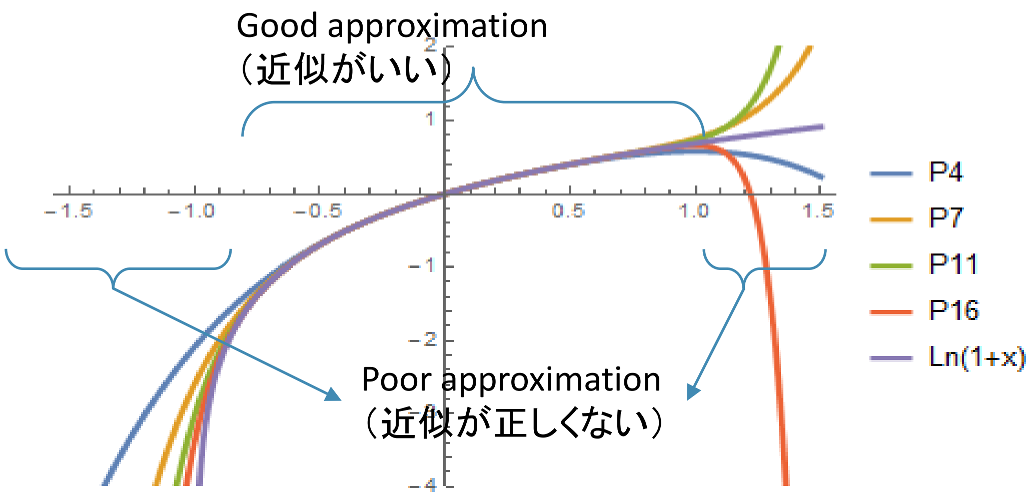


Some usual Taylor expansions

function	Taylor expansion at $x = 0$	Neighbor of $x = 0$
e^x	$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$	$x \in \mathbb{R}$
$\sin(x)$	$\sin(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + (-1)^n \frac{x^{2n+1}}{2n+1} + \cdots$	$x \in \mathbb{R}$
$\cos(x)$	$1 - \frac{x^2}{2} + \frac{x^4}{4} + \cdots + \frac{(-1)^n x^{2n}}{(2n)!} + \cdots$	$x \in \mathbb{R}$
$\ln(1+x)$	$x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + \frac{(-1)^{n-1} x^n}{n} + \cdots$	$-1 < x \leq 1$
$\sqrt{1+x}$	$\sum_{k=0} \frac{(-1)^n (2n)!}{(1-2n)(n!)^2(4^n)} x^n = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \frac{5x^4}{128} + \cdots$	$-1 < x < 1$

Comments

1) Be careful about "neighborhood" in table page 11.
This is related to convergence of "power series"（ベキ級数）



$\ln(1+x)$ is valid for $|x| < 1$.
If $|x| > 1$ the Taylor polynomials are NG

2) In practice: We must be able to compute $f(x_0), f'(x_0), f''(x_0), \ldots$

Example: $\cos\left(\frac{7}{5}\right)$ cannot be computed easily…

→…Choose a point x_0 close to $x = \frac{7}{5}$ so that $\cos(x_0)$ is known.

$x_0 = \frac{\pi}{3}$ is good because: $\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$ and $\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$

Computing limits with Taylor’s expansions （テイラー展開で極限をとる）

- **Example**: $\lim_{x \to 1} \frac{\ln(x)}{x-1}$

Indeterminate form of type "0/0".

- **Solution:** $\ln(x) = (x-1) - \frac{1}{2}(x-1)^2 + \epsilon_2\,(x-1)$ if $|x| < 1$

Thus $\frac{\ln(x)}{x-1} = 1 - \frac{1}{2}(x-1) + \frac{\epsilon_2(x-1)}{x-1}$ if $|x| < 1$.

Therefore $\lim_{x \to 1} \frac{\ln(x)}{x-1} = 1$.

Exercise

- Compute the following limits:

1. $\lim_{x \to 0} \frac{e^x - (1+x)}{x^2} =$

2. $\lim_{x \to 0} \frac{\ln\left(1+x^2\right)}{1-\cos(x)} =$

Program L4-3

1. Approximating functions by polynomials:
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2. Approximating zeroes of functions:
   Newton’s method
   関数の零を近似する:ニュートン法

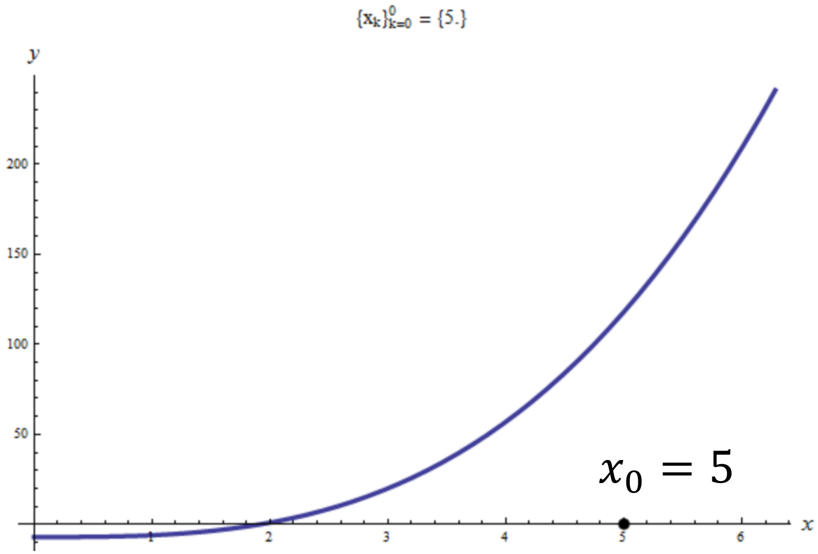
Newton's method 1: principle
ニュートン法:原理

- Example: Find the zero of $f: x \mapsto x^3 - 7$ (approximate $\sqrt[3]{7}$)

$\sqrt[3]{7} \approx 1.912931183$

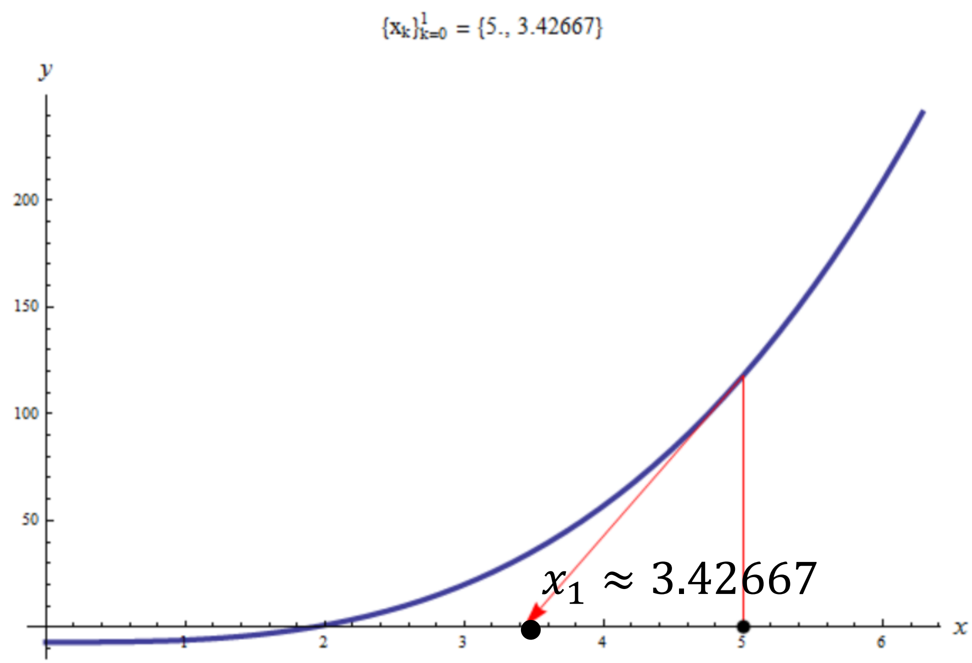
Choose an initial value x_0 (初期値）

Here we choose $x_0 = 5$

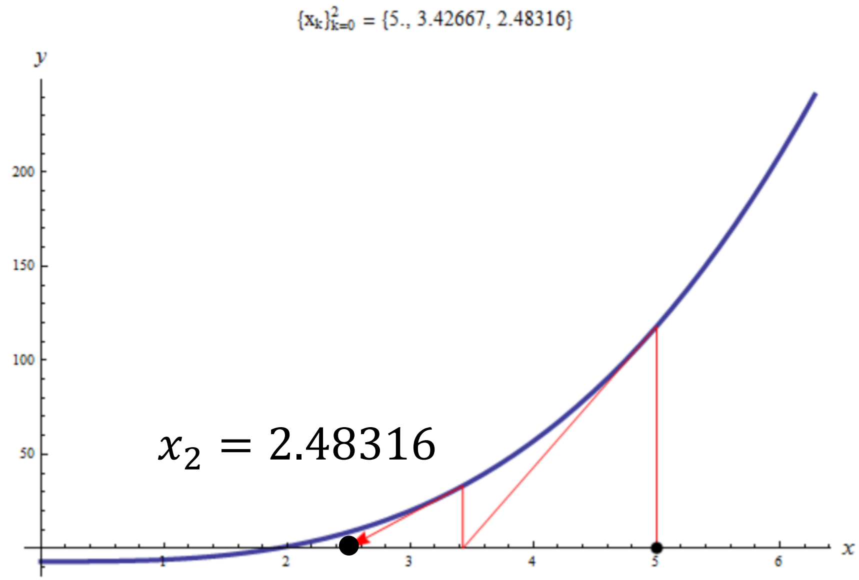


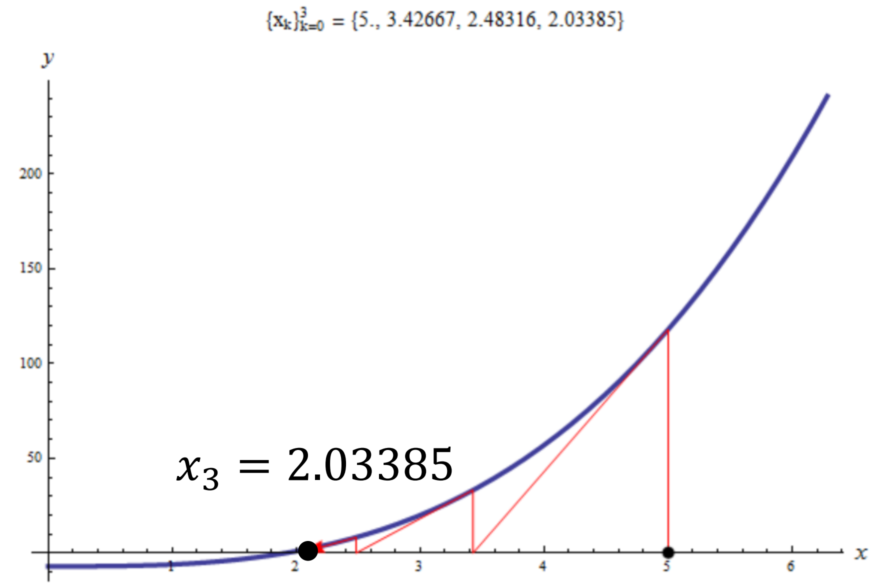
Follow the tangent $y = T(x)$ of f at x_0 and find the intersection with $y = 0$.

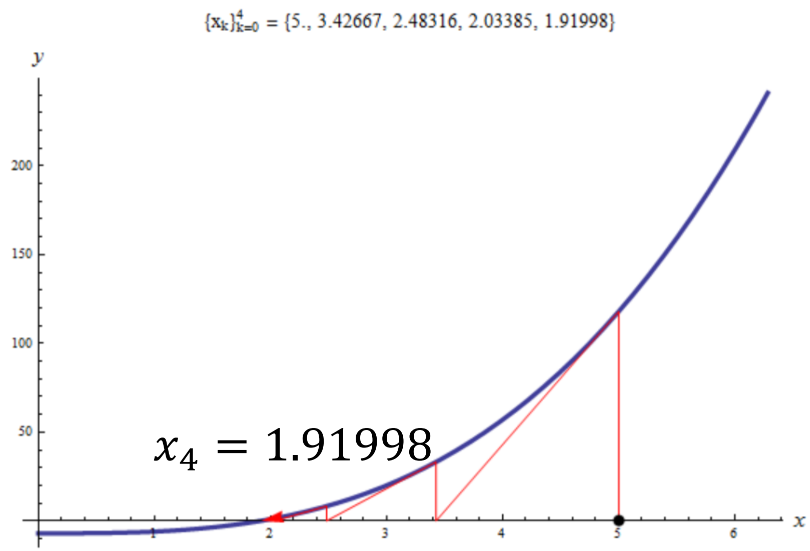
x_0におけるfの接線$y = T(x)$ に沿って、線$y = 0$との交点x_1を計算する。



Newton's method 2: 2nd to 4th iterations(反復)



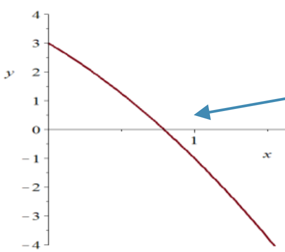




Iteration k（反復）	Error (誤差） $|\sqrt[3]{7} - x_k|$
0	3.87
1	1.514
2	0.57
3	0.12
4	0.007

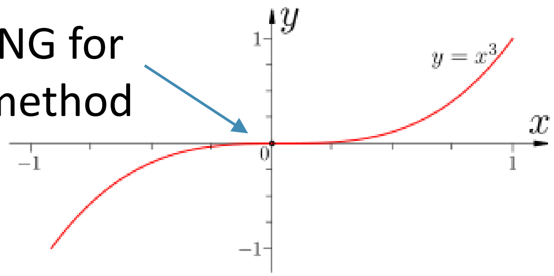
Formula for Newton's method
ニュートン法における数式

- The function must be increasing or decreasing around the zero z, $f'(z) \neq 0$.



$f'(0) < 0$ OK for Newton's method

$f'(0) = 0$ NG for Newton's method



An initial point x_0 must be selected (possibly near to the zero z).

To compute x_1: The equation of the tangent is

$$y = f'(x_0)x + f(x_0) - f'(x_0)x_0$$

If $y = 0$, we find that

$$f'(x_0)x + f(x_0) - f'(x_0)x_0 = 0,$$

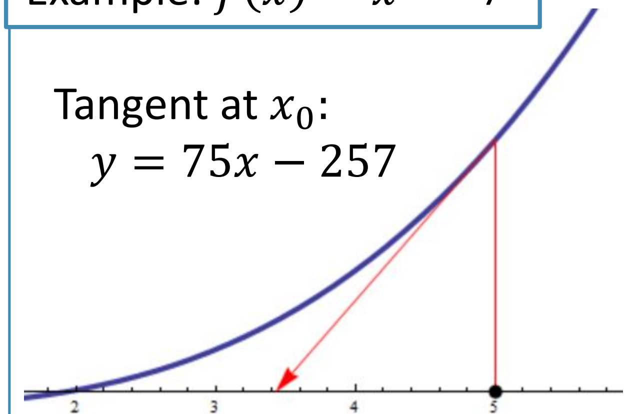
And $x_1 = \frac{f'(x_0)x_0 - f(x_0)}{f'(x_0)} = \boxed{x_0 - \frac{f(x_0)}{f'(x_0)} = x_1.}$

注意: $f'(x_0) \neq 0$. If $f'(x_0) = 0$ choose another initial value x_0

Example: $f(x) = x^3 - 7$

Tangent at x_0:

$y = 75x - 257$



$x_1 = \frac{257}{75}$ $x_0 = 5$

Formula for Newton's method
ニュートン法における数式

To compute x_2: The equation of the tangent is

$$y = f'(x_1)x + f(x_1) - f'(x_1)x_1$$

If $y = 0$, we find that

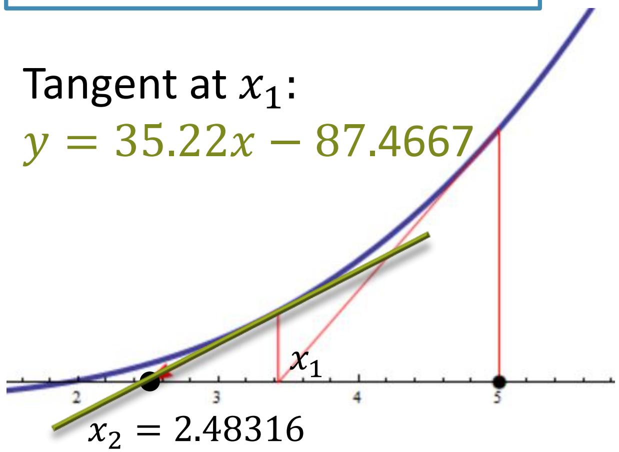
$$f'(x_1)x + f(x_1) - f'(x_1)x_1 = 0,$$

And $x_2 = \frac{f'(x_1)x_1 - f(x_1)}{f'(x_1)} = \boxed{x_1 - \frac{f(x_1)}{f'(x_1)} = x_2.}$

注意: $f'(x_1) \neq 0$. If $f'(x_1) = 0$ choose another initial point x_0 and start over.

Example: $f(x) = x^3 - 7$

Tangent at x_1:
$y = 35.22x - 87.4667$



Generally:
If $(x_0, x_1, \ldots, x_k)$ have been computed. The formula to compute x_{k+1} is:

$$\boxed{x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}}$$ (assuming that $f'(x_k) \neq 0$)

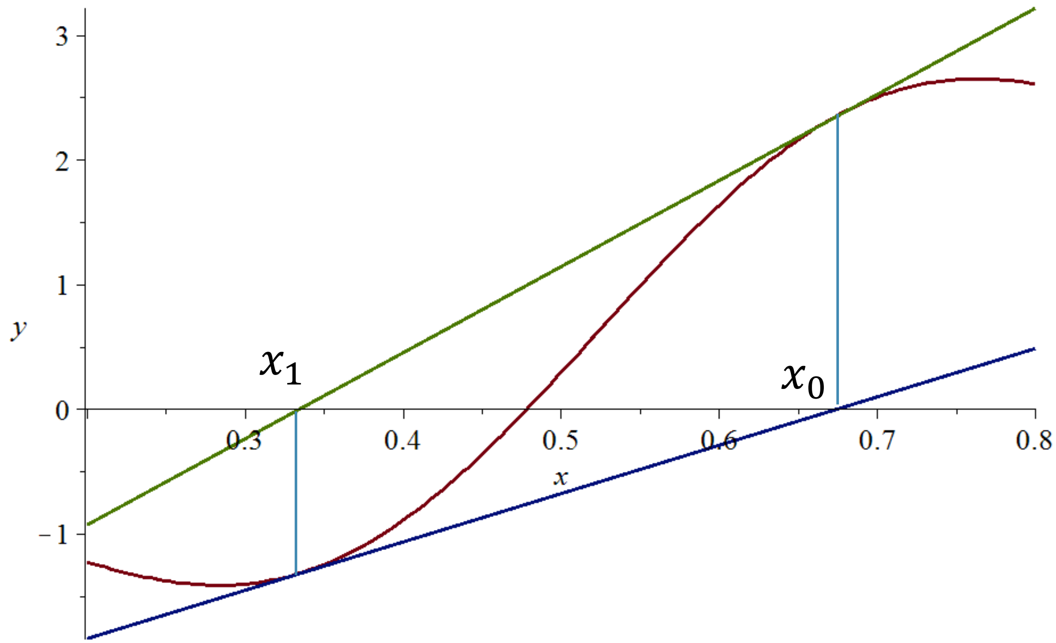
Comments on Newton's method
ニュートン法に関するコメント

- The construction of the sequence of approximate numbers $(x_0, \dots, x_n, \dots)$ is iterative (反復法）
  The whole sequence depends on x_0.
  (すべての例数は、初期値x_0 にだけ依存する）

- Choosing a good x_0 is very important:
  bad choice of x_0 means that the method:
  1. fails: we have $f'(x_i) = 0$ for some i
  2. is very slow $(f'(x_i) \approx 0)$ for some i
  3. finds another zero z' that the zero that we were looking for.

- If x_0 is well-chosen the method is very efficient:
  usually $n = 5$ or 6 iterations give a very good approximation.

Bad choice of initial value x_0

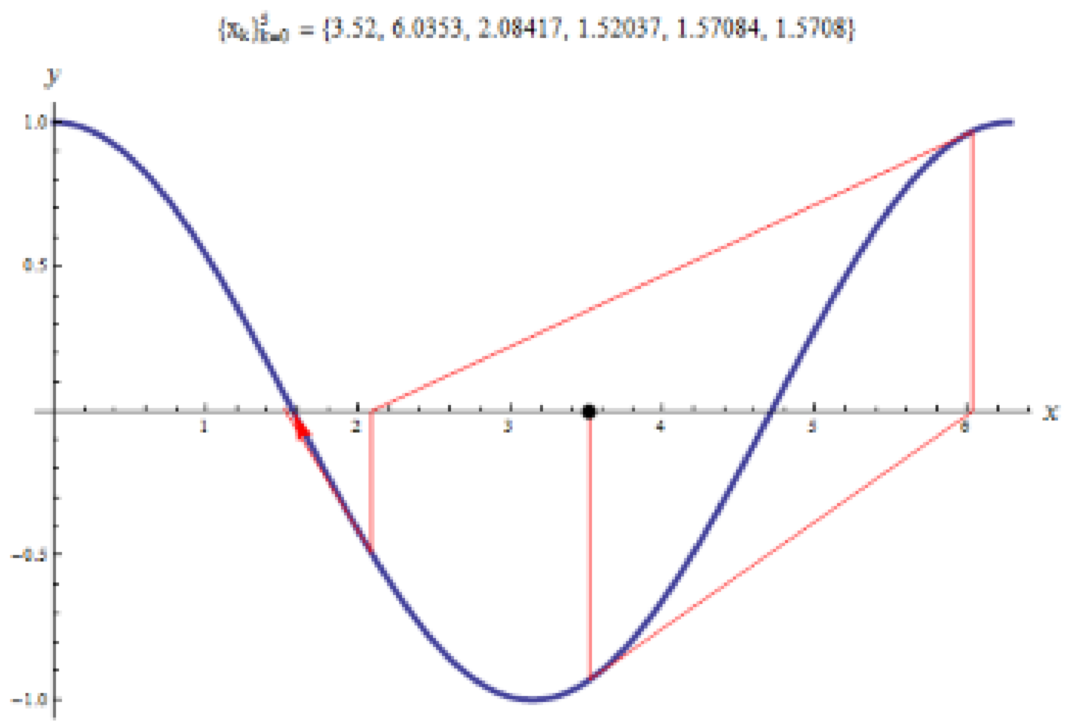


Here the method repeats infinitely:

$x_0 = x_2 = x_4 = \cdots \qquad x_1 = x_3 = x_5 \cdots$

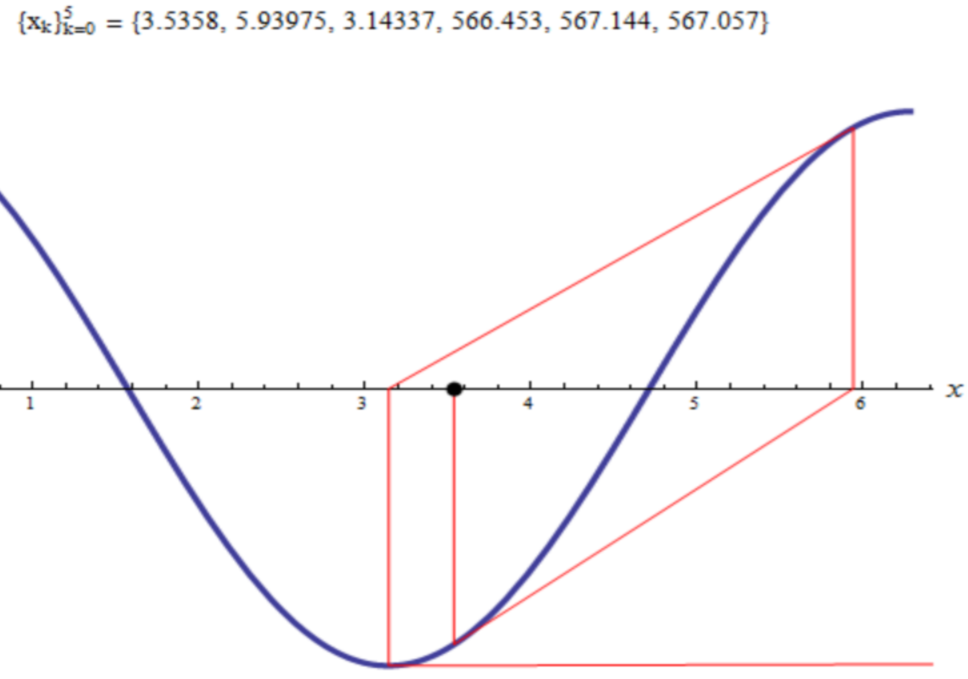
This almost never happens！（ほとんどあまり起こらない）

Bad choice of initial value x_0 (II)



The method finds a solution but not the one expected !
解法は解を見つけているが、期待された解がではない。

Bad choice of initial value x_0 (III)



The method falls on a point x_i where $f'(x_i) = 0$. That doesn't work!
This problem almost never occurs.
この問題はほとんどあまり起こらない。

Convergence to a zero of f
関数fの零に収束

Theorem:

Let $f: E \subset \mathbb{R} \to \mathbb{R}$, and $z \in E$, such that $f(z) = 0$.

Assume that on an interval $(a, b) \subset E$, $z \in (a, b)$, f is differentiable（微分可能）.

Let $x_0 \in (a, b)$ be “near enough” to z.

The sequence $(x_0, x_1, x_2, \ldots, x_k, \ldots)$ converges（収束する）to z, where $x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$.

$$\lim_{k \to \infty} |x_k - z| = 0$$

- How fast does it converge ?どのくらい速さで収束するか？

Rate of convergence (收束率)

- Let the error (誤差) be $\epsilon_k = |z - x_k|$ after the k-th iteration（第k反復で得られた値x_k）. Assume that $f'(x_k)$ is not close to 0 :

$$\epsilon_{k+1} = |z - x_{k+1}| \le M|\epsilon_k|^2$$

Rate of convergence is quadratic (or quadratic convergence). （二次収束）

- **Proof**: By Taylor's 1st order approximation (page 6):
There is d close to x_k such that

$$f(z) = f(x_k) + f'(x_k)(z - x_k) + \frac{1}{2} f''(d)(z - x_k)^2$$

$$\frac{f(x_k)}{f'(x_k)} + (z - x_k) = \frac{-f''(d)}{2f'(x_k)}(z - x_k)^2$$

- By definition of Newton's iteration: $z - x_{k+1} = \frac{-f''(d)}{2f'(x_k)}\underbrace{(z - x_k)^2}_{\epsilon_k}$

- $\epsilon_{k+1} = \left|\frac{-f''(d)}{2f'(x_k)}\right| \epsilon_k^2$. The term $\left|\frac{-f''(d)}{2f'(x_k)}\right| \le M \Rightarrow |\epsilon_{k+1}| \le M|\epsilon_k|^2$