Essential Mathematics for Global Leaders I

Lecture 4-2

Differentiation II

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Plan (tentative)

[4/13] L1 : introduction. Review of high-school mathematics in English.

[4/20-27] L2-3 : Functions and graphs. Plotting with Mathematica I (グラフをプロットする)

[5/7] L4 : Infinitely small and large : limits (極限)

[5/11] L5 : Differentiation (連続性と微分法)

[5/18] L6 : Differentiation II : extrema, related rates ...（極値と...）

[5/25] L7 : Differentiation III : Newton's method, Taylor's expansion
(ニュートン法とテイラー展開）

[6/1] L8 : Mid-term test. Integration I : definition, fundamental theorem of calculus　積分I.

[6/8] L9 : computation of indefinite integrals　不定積分

[6/15] L10 : Application of Integration I : length, volume and surfaces
積分の応用：長さ、面積、体積

[6/22] L11 : Application of Integration II : average, center of mass （質量中心）, work of a force.

[6/29] L12 : Ordinary Differential equations (one variable)　常微分方程式

[7/6] L13 : Linear Differential Equations of order 2 : harmonic oscillators (small-angle pendulum, spring). 二階線形微分方程式：調和振動子　(振幅が小さい振り子、ばね)

[7/13] L14 : Ordinary Differential Equations with Mathematica. Mathematicaを利用して常微分

Program L4-2

1. Mean Value Theorem And Extrema
   平均値の定理と極値

2. Compared growth of Exp, Ln, x^n at ∞

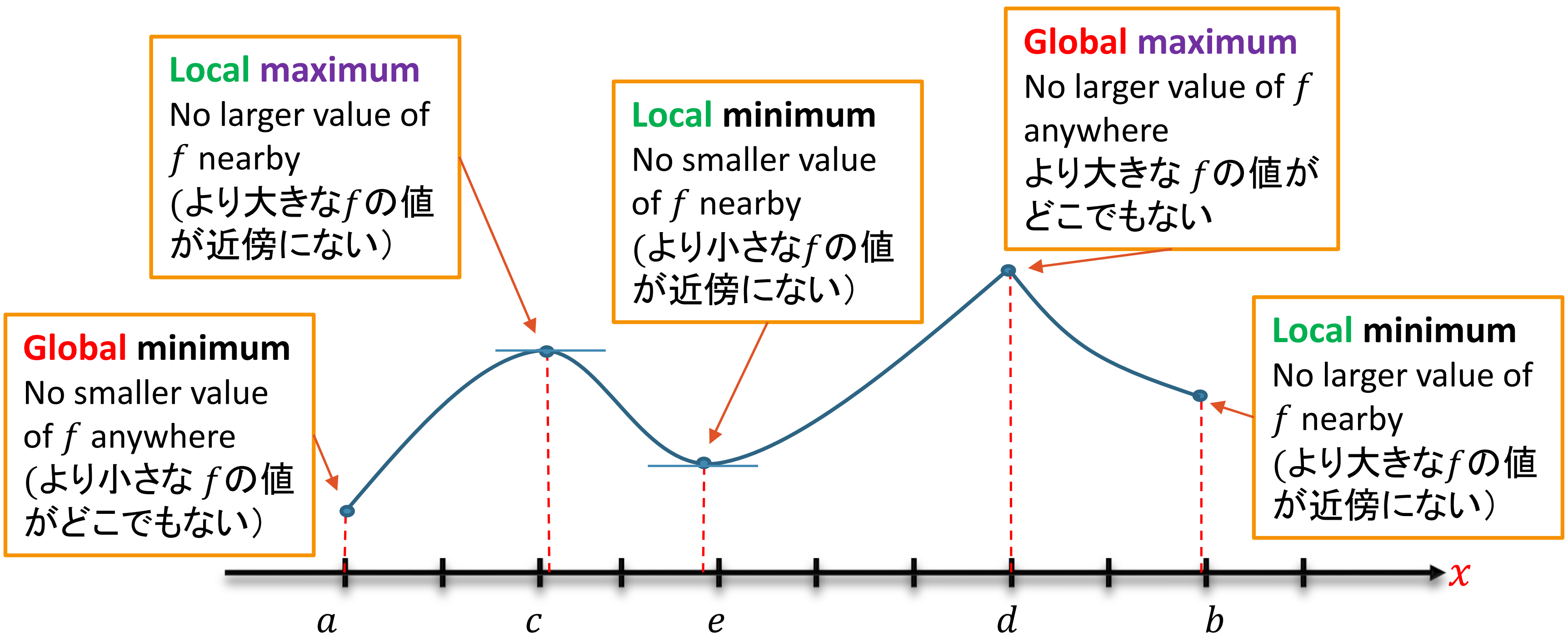
Extrema (極値)

- **Local – Global maximum**（局所・大域の極小・極大） $f: E \subset \mathbb{R} \to \mathbb{R}$

  If $f(x) \leq f(c)$ for all $x \in E$ then $f(c)$ is a global maximum.

  If there exists $\alpha, \beta \in E$, $\alpha < c < \beta$, such that:

  $f(x) \leq f(c)$ for all $x \in (\alpha, \beta)$ then $f(c)$ is a local maximum.



Finding Extrema (I) First derivative test （極値をとる: 一階導関数の判定）

Theorem: If $f: E \subset \mathbb{R} \to \mathbb{R}$ has a local maximum or minimum at an interior point $c \in E$, and if f is differentiable at c then:
もしfは内点cにおいて局所の極大or極小があれば、かつcにおいて微分可能ならば、

$$f'(c) = 0$$

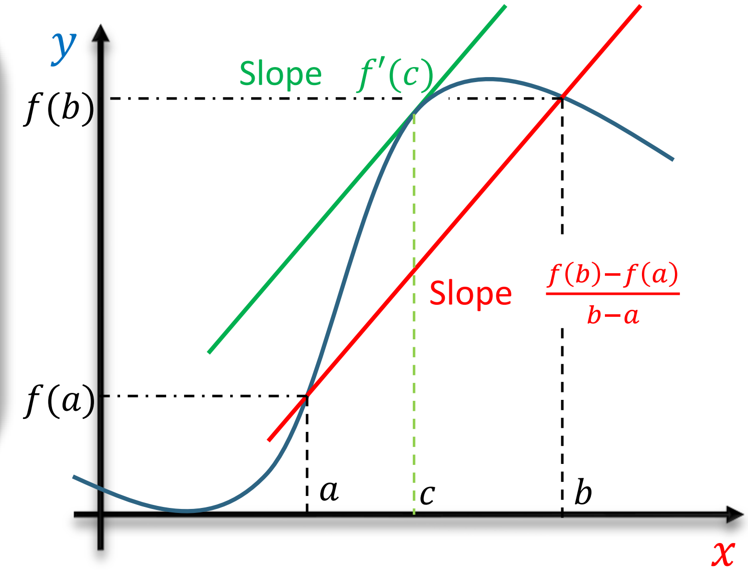
注意点

- If c is an endpoint this is not true (→ point b in the graph page ??: local minimum but $f'(b) \neq 0$)
  もしもcは端点ならば、$f'(c) = 0$が成り立つとはかぎらない。
- If f is *not* differentiable at c, then $f(c)$ may be a maximum or minimum anyway (point d in the graph page ??)
  もしfは点cにおいて微分可能でないと言っても、$f(c)$は極小or極大であるかもしれない。
- The proof is easy (証明も簡単)

Mean Value Theorem (平均値の定理)

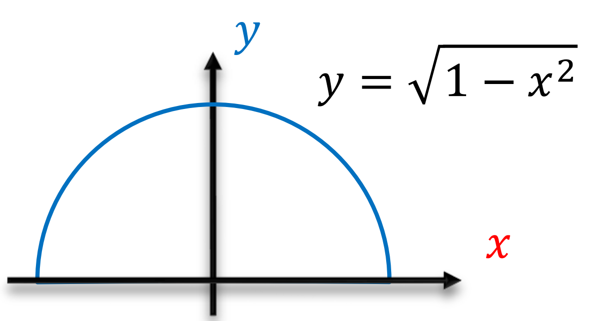
Theorem (MVT): Let $f: [a, b] \to \mathbb{R}$ be a continuous function, differentiable on (a, b). There exists at least one point c such that:

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$



- Exercise: Let $f: [-1,1] \to \mathbb{R}$, $x \mapsto \sqrt{1 - x^2}$. Find a point c such that:

  $$f'(c) = f(1) - f(-1)/1 - (-1)$$



Answer: $x \to \sqrt{1 - x^2}$ is *not* differentiable at $a = -1$ nor at $b = 1$, but is differentiable on the interval $(-1; 1)$ so the theorem applies.

We have $c = 0$, indeed:

$$\frac{f(b) - f(a)}{b - a} = \frac{f(1) - f(-1)}{1 + 1} = 0 = f'(0) = f'(c)$$

Corollary1: Constant function
系1(=直接の結果):定値写像

Corollary 1(系): If $f'(x) = 0$ for all $x \in (a,b)$, then $f(x) = C \in \mathbb{R}$ on (a,b).

- Proof（証明）: Let $x_1 < x_2$ in the interval (a,b). By MVT, there is a $c \in (x_1, x_2)$ such that $\frac{f(x_2)-f(x_1)}{x_2-x_1} = f'(c)$.

Therefore $\frac{f(x_2)-f(x_1)}{x_2-x_1} = 0 \Rightarrow f(x_1) = f(x_2)$ □

Corollary: If $f'(x) = g'(x)$ on an interval (a,b) then $f(x) = g(x) + C$ for a constant（定数）$C \in \mathbb{R}$.

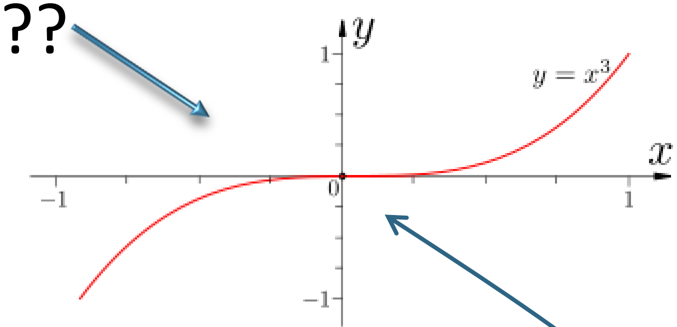
Test for increasing and decreasing function

Corollary2: $f:[a,b]\to\mathbb{R}$ continuous, and differentiable on (a,b).
If $f'(x)>0$ on (a,b) then f is increasing (増加) on $[a,b]$.
If $f'(x)<0$ on (a,b) then f is decreasing (減少) on $[a,b]$.

- **Proof** (証明): Let $x_1<x_2$ in $[a,b]$. By the MVT, there is a $c\in(x_1,x_2)$ such that $f(x_2)-f(x_1)=f'(c)(x_2-x_1)$. If $f'(c)>0$ then $f(x_2)>f(x_1)$, and thus f is increasing. $\square$

まとめ: Derivative tests for local extrema
一階、二階の導関数による極値点の判定

- $f: E \to \mathbb{R}$ differentiable twice (導関数も微分可能).
- **1st derivative test**: find the set of critical points (臨界点） $C := \{x \in E : f'(x) = 0\}$
- For $c \in C$: Maybe it is a
  - local minimum
  - local maximum
  - ???



- **2nd derivative test**:
  - $f''(c) > 0 \to$ local minimum (局所の極小）
  - $f''(c) < 0 \to$ local maximum （局所の極大）
  - $f''(c) = 0 \to$???

Exercise: finding extrema（極値の探求）

1. Tell where the following functions are increasing and decreasing.
2. Find the local & global extrema
   a. $g(t) = -t^2 - 3t + 3$

      $g'(t) = -2t - 3$. g' is >0 on $(-\infty; -\frac{3}{2})$ and <0 on $\left(-\frac{3}{2}; +\infty\right)$ therefore g is ↗ $(-\infty; -\frac{3}{2})$ and ↘ on $\left(-\frac{3}{2}; +\infty\right)$
      Global maximum at $t = -\frac{3}{2}$.

   b. $f(r) = r^3 + 16r$ $\quad f'(r) = 3r^2 + 16 > 0$ for any $r \in \mathbb{R}$
      therefore f is ↗ and there is no max nor min.

   c. $f(x) = \frac{x^2-3}{x-2}, x \neq 2$ $\quad f'(x) = \frac{2x(x-2)-(x^2-3)}{(x-2)^2} = \frac{x^2-4x+3}{(x-2)^2}$
      $\Delta = 16 - 4 \times 3 = 4$ so $x^2 - 4x + 3 = (x-1)(x-3)$.
      Thus $f'(x) < 0$ for $1 < x < 3$ and $f'(x) > 0$ elsewhere.
      $f'(x) = 0$ iff $x = 1$ or 3. Both are local maximum. Since $f(1) = 2$ and $f(3) = 6$ it follows that 3 is the global maximum

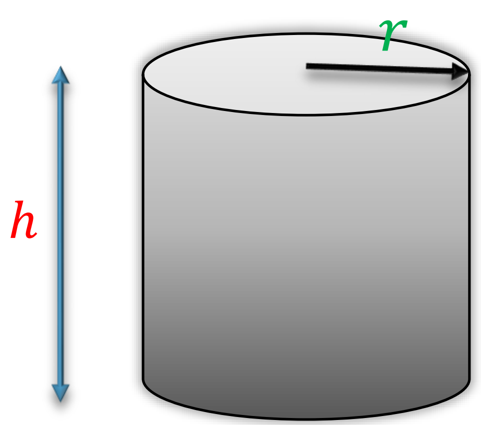
Application: optimization
（適用：制約付き最適化）

- Kind of "problem with constraint" （制約付き問題）

$$\min_{g(x)=0} f(x)$$

Find the minimum of a function f that must satisfy also the constraint （制約） $g(x) = 0$.

- Example:



Given a can（缶, = a cylinder 円筒）

Constraint: Volume=$h\pi r^2$=1000cm^3

Find h and r that minimizes the surface (面積= $S(r,h)$)of the can:

$$\min_{h\pi r^2=1000} S(r,h)$$

Answer: $r = \sqrt[3]{\frac{500}{\pi}} \approx 5.42cm \quad h = 2r \approx 10.84cm$

Program L4-2

1. Mean Value Theorem And Extrema
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2. Compared growth of Exp, Ln, x^n at ∞

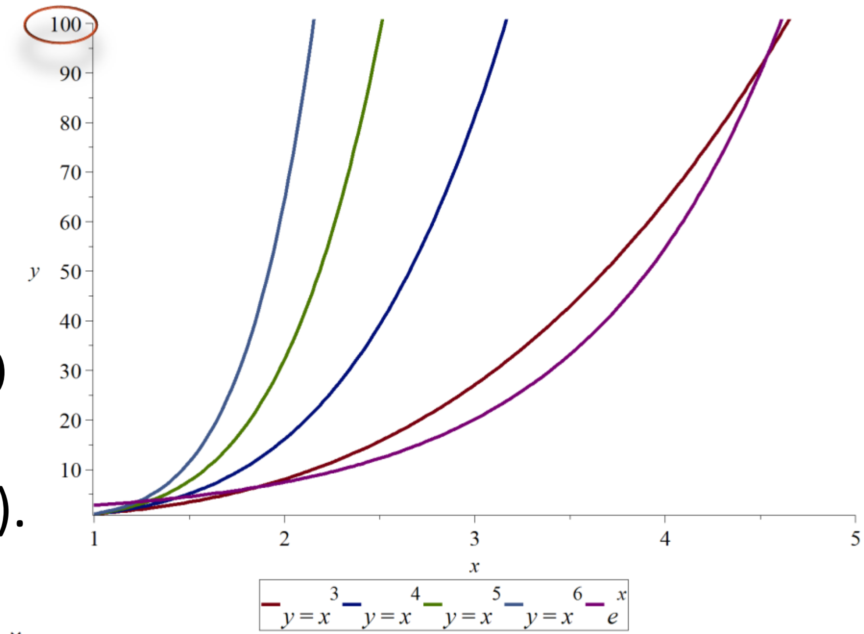
Asymptotic and growth (I)

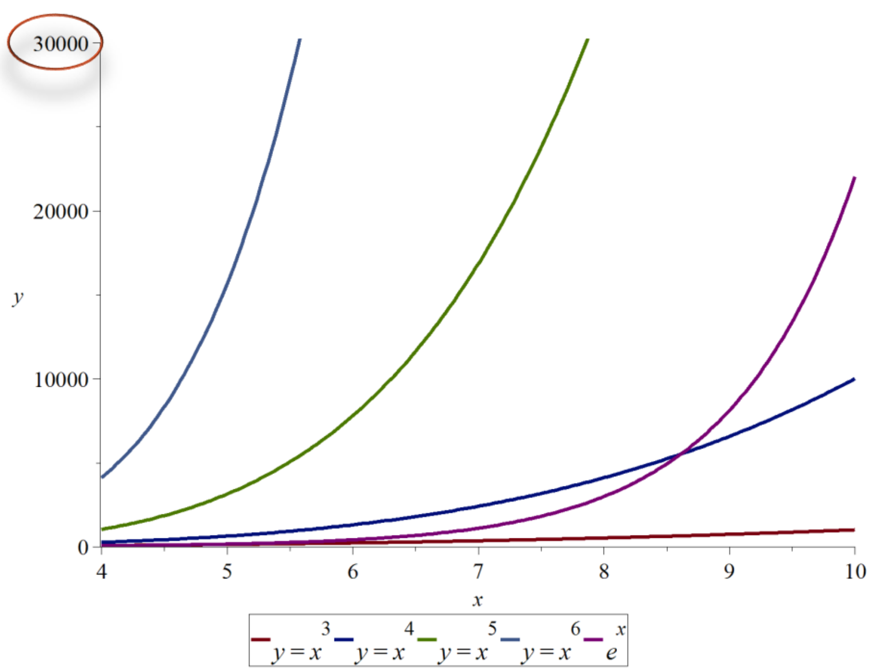
We want to compute:

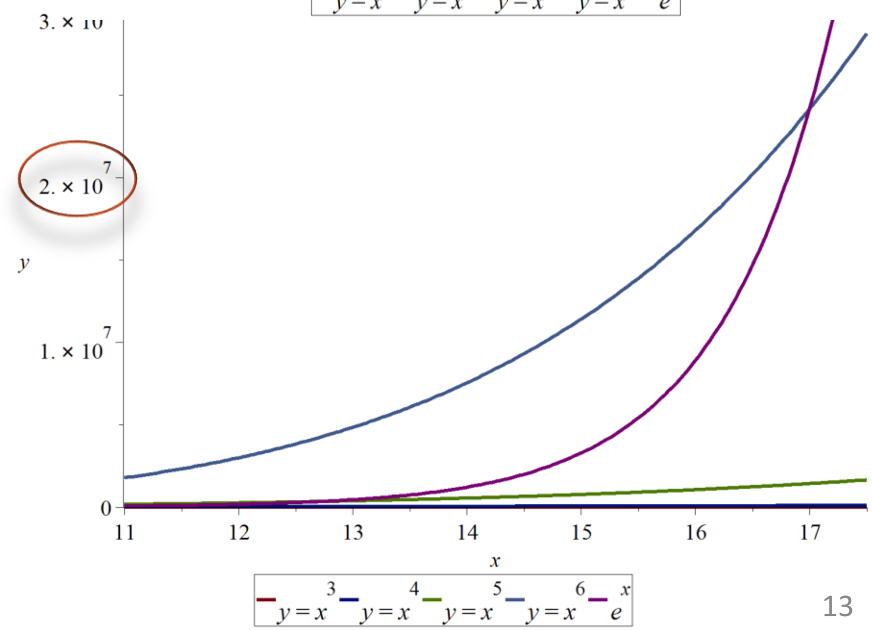
$$\lim_{x\to\infty} \frac{e^x}{x^n} \quad \text{(ex: } \lim_{x\to\infty} \frac{e^x}{x^{100}} = ?\text{)}$$

$$\lim_{x\to\infty} \frac{(\ln(x))^n}{x} \quad \text{(ex: } \lim_{x\to\infty} \frac{(\ln(x))^{100}}{x} = ?\text{)}$$

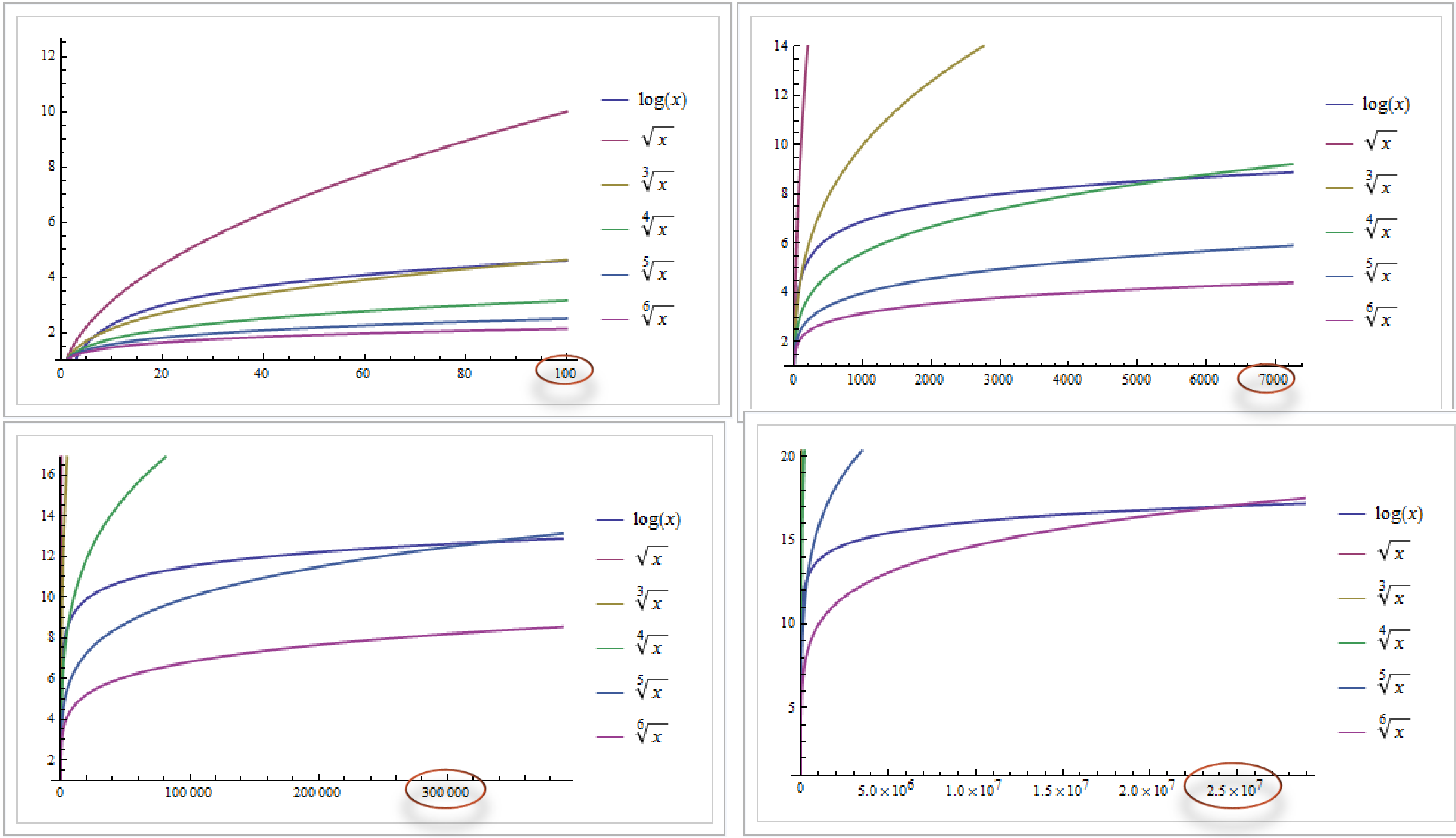
(indeterminate form of type ∞/∞).







- From the graphs, it seems that $\lim_{x\to\infty} \frac{e^x}{x^n} = +\infty$ whatever is $n > 0$.
- What about $\lim_{x\to\infty} \frac{\ln(x)}{x^{1/n}} = \lim_{x\to\infty} x^n \ln(x)$ when $n > 0$?



Asymptotic and growth (III)

From the graphs, it seems that $\lim_{x\to\infty} \frac{\ln(x)}{x^{1/n}} = 0$ whatever is $n > 0$.

直観を超える： 証明 Prove that $\lim_{x\to\infty} \frac{e^x}{x^n} = \infty$ for $n > 0$.

1. Show that $e^x \geq (n+1)x$ for $x \in (n, \infty)$
(Hint: Let $f(x) = e^x - (n+1)x$. Is f increasing ?)

Answer: $f'(x) = e^x - (n+1)$.
$f'(x) > 0$ (sure if $x > n$ because $e \approx 2.78$)
Therefore f is ↗ on (n, ∞).
Next, we have $f(n) > 0$ (If $n = 1$ then $f(1) = e - 2 > 0$ and clear if $n \geq 2$)
Therefore $e^x > (n+1)x$ for $x \in (n, \infty)$.

2. Deduce that $e^x \geq x^{n+1}$ for $x > n(n+1)$.

Answer: Let $y = x/n + 1$, $and\ y > n(\Rightarrow x > n(n+1))$.

By 1), $e^y > (n+1)y$, equivalently $e^{\frac{x}{n+1}} > x$. Thus $\left(e^{\frac{x}{n+1}}\right)^{n+1} > x^{n+1}$
$\Rightarrow e^x > x^{n+1}$ for $x > n(n+1)$

3. Use the sandwich theorem to compute the limit.
Answer: By 2), $\frac{e^x}{x^n} > x$ for $x > n(n+1)$. Since $\lim_{x\to\infty} x = \infty$ the sandwich theorem (Lect. 3 page 22) implies that $\lim_{x\to\infty} \frac{e^x}{x^n} = \infty$.

Asymptotic and growth (IV)

- Deduce the following limit for $n, m \in \mathbb{Z}_{>0}$

(Hint: Use the limit of the previous page and consider a change of variable. 前のページの極限を利用して変数の変換を考えなさい）

1. $\lim_{x \to \infty} \frac{(\ln(x))^n}{x}$ *Hint:* $y = \ln(x)$. *Answer is 0*

2. $\lim_{x \to 0^+} x|\ln(x)|^n$ *Hint:* $y = \ln\left(\frac{1}{x}\right)$. *Answer is 0*

3. $\lim_{x \to -\infty} |x|^n e^{mx}$ *Hint:* $y = e^{-mx}$. *Answer depnds.*

Homework (II)

A rectangle is inscribed in a semicircle of radius 2.半径2の半円に内設させる長方形がある。
What is the largest area the rectangle can have, and what are its dimension ?
最も広い面積を持つ長方形が何か。