Essential Mathematics for Global Leaders I

Lecture 4

Differentiation I

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Plan (tentative)

[4/13] L1 : introduction. Review of high-school mathematics in English.

[4/20-27] L2-3 : Functions and graphs. Plotting with Mathematica I (グラフをプロットする)

[5/7] L4 : Infinitely small and large : limits (極限)

[5/11] L5 : Differentiation (連続性と微分法)

[5/18] L6 : Differentiation II : extrema, related rates ...（極値と...）

[5/25] L7 : Differentiation III : Newton's method, Taylor's expansion (ニュートン法とテイラー展開）

[6/1] L8 : Mid-term test. Integration I : definition, fundamental theorem of calculus 積分I.

[6/8] L9 : computation of indefinite integrals 不定積分

[6/15] L10 : Application of Integration I : length, volume and surfaces
積分の応用：長さ、面積、体積

[6/22] L11 : Application of Integration II : average, center of mass （質量中心）, work of a force.

[6/29] L12 : Ordinary Differential equations (one variable) 常微分方程式

[7/6] L13 : Linear Differential Equations of order 2 : harmonic oscillators (small-angle pendulum, spring). 二階線形微分方程式：調和振動子 (振幅が小さい振り子、ばね)

[7/13] L14 : Ordinary Differential Equations with Mathematica. Mathematicaを利用して常微分

Program

1. Definition, Introduction （定義）

2. Motion. Chain rule & rates of change (運動。連鎖~~法~~律、変化率）

Tangent line to a curve graph（曲線の接線）

Let $f: E \subset \mathbb{R} \to \mathbb{R}$ a function, $x_0 \in E$.

If $\lim_{h \to 0} \frac{f(x_0+h)-f(x_0)}{h} = m$ exists and is finite (存在して有限であれば)

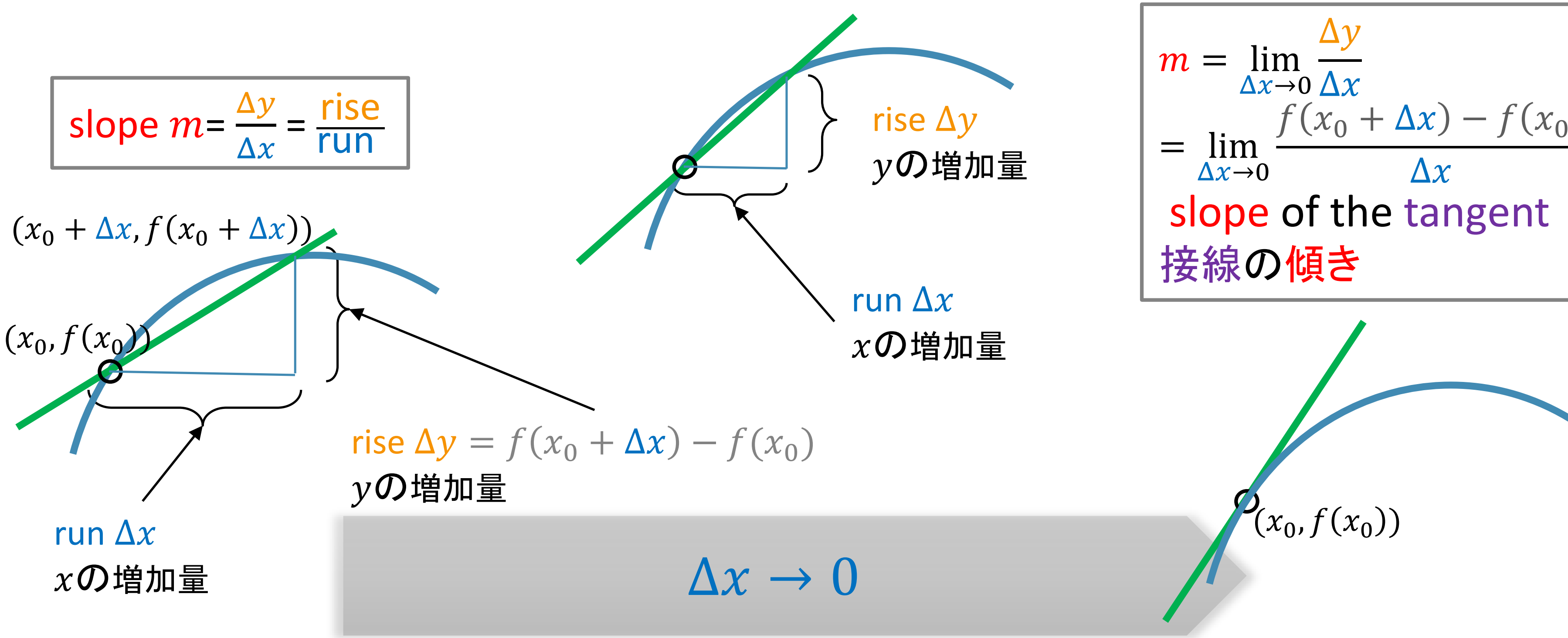
m is called the slope of the tangent line to the curve $y = f(x)$ at the point x_0, (曲線 $y = f(x_0)$上のx_0における接線の傾き).

slope $m = \frac{\Delta y}{\Delta x} = \frac{\text{rise}}{\text{run}}$

$$m = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

slope of the tangent

接線の傾き



Derivative of a function (導関数)

- $f: E \subset \mathbb{R} \to \mathbb{R}$, is differentiable at $x_0 \in E$（x_0において微分可能）, if the limit $\lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0}$ exists (and is not $\pm\infty$).
  The limit is denoted $f'(x_0)$ and is called the derivative of f at the point x_0（$f'(x_0)$を $x = x_0$において微分係数と呼ぶ）。

- Let $F \subset E$ be the set of points where f is differentiable (微分可能になるEの部分集合をFとする）、
  $$f': F \to \mathbb{R}, \qquad x \mapsto f'(x)$$
  is the derivative function of f （fの導関数と呼ぶ）.
  If $F = E$, then we say that f is differentiable.

- (用語)To differentiate function f: 関数fを微分する
  Differentiation: 微分法
  Another notation（他の記号）: $f'(x) = \frac{d}{dx} f(x)$ (Reading: "d f d x")

- **One-sided derivative (片側導関数)**

$$\lim_{h \to 0^+} \frac{f(x_0+h)-f(x_0)}{h}$$

right-hand derivative 右微分

$$\lim_{h \to 0^-} \frac{f(x_0+h)-f(x_0)}{h}$$

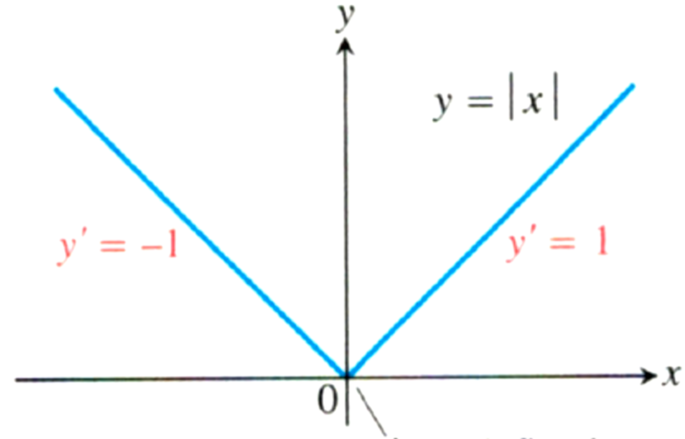
left-hand derivative 左微分

- Lecture 3, page 11:
  $x_0 \in E$ is interior（内点）

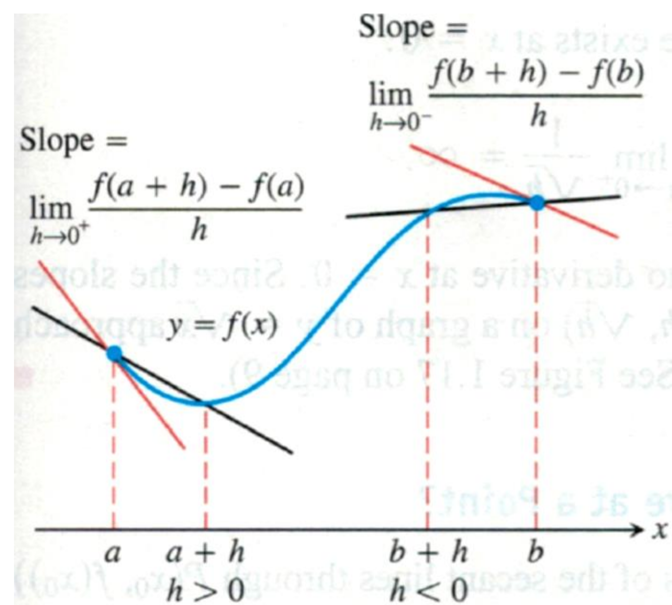
f differentiable at x_0 ⇔ right an left-hand derivative exist, $\neq \pm\infty$, and are equal.

- $x_0 \in E$ is an endpoint（端点）say a left endpoint.

f differentiable at x_0 ⇔ left-hand derivative exist, $\neq \pm\infty$



y' not defined at $x = 0$ (interior)
Right-hand derivative ≠ left-hand derivative



Derivatives at endpoints are one-sided limits

Compute the derivative of the functions f_1 & f_2

- $f_1 : x \mapsto x^n$

$$A_n(h) := \frac{f_1(x+h) - f_1(x)}{h} = \frac{(x+h)^n - x^n}{h}$$

$$= \frac{x^n + nhx^{n-1} + h^2\left(\frac{(n-1)n}{2}x^{n-2} + \cdots + C_n^k h^{k-2}x^{n-k} + \cdots + h^{n-2}\right) - x^n}{h} =$$

$$nx^{n-1} + hB_n$$

$\lim_{h\to 0} A_n(h) = nx^{n-1}$ therefore $f_1'(x) = nx^{n-1}$.

- $f_2 : x \mapsto \frac{1}{x^n}$

$$A_n(h) = \frac{1/(x+h)^n - 1/x^n}{h} = \frac{x^n - (x+h)^n}{h(x+h)^n x^n}$$

$$= \frac{x^n - x^n - nhx^{n-1} - h^2 B_n}{h(x+h)^n x^n} = \frac{-nx^{n-1}}{(x+h)^n x^n} - \frac{hB_n}{(x+h)^n x^n}$$

Since $\lim_{h\to 0} \frac{hB_n}{(x+h)^n x^n} = 0$, and $\lim_{h\to 0} \frac{-nx^{n-1}}{(x+h)^n x^n} = \frac{-nx^{n-1}}{x^{2n}} = \frac{-n}{x^{n+1}}$. Thus $f_2'(x) = \frac{-n}{x^{n+1}}$

Compute the derivative functions of f_3 & f_4

- $f_3 : x \mapsto e^x$

$$A_x(h) \coloneqq \frac{e^{x+h} - e^x}{h} = e^x \frac{e^h - 1}{h}$$

$\lim_{h \to 0} A_x(h) = e^x$, thus $f_3'(x) = e^x$.

- $f_4 : x \mapsto \ln(x)$

$$A_x(h) \coloneqq \frac{\ln(x+h) - \ln(x)}{h} = \frac{1}{x} \cdot \frac{\ln(1 + h/x)}{h/x}$$

But $\lim_{h \to 0} A_x(h) = \frac{1}{x} \lim_{h' \to 0} \frac{\ln(1+h')}{h'}$ with $h' = \frac{h}{x}$.

Therefore, $f_4'(x) = \frac{1}{x} \lim_{h' \to 0} \frac{\ln(1+h')}{h'} = \frac{1}{x}$.

Compute the derivative functions of: f_6 & f_7

- $f_6 : x \mapsto \sin(x)$

$$A_x(h) \coloneqq \frac{\sin(x+h) - \sin(x)}{h}$$

$$= \frac{\sin(x)\cos(h) + \cos(x)\sin(h) - \sin(x)}{h}$$

$$= \sin(x)\left(\frac{\cos(h) - 1}{h}\right) + \cos(x)\left(\frac{\sin(h)}{h}\right)$$

Since $\lim_{h\to 0} \frac{\sin(h)}{h} = 1$ and $\lim_{h\to 0} \frac{\cos(h)-1}{h} = 0$,

$\lim_{h\to 0} A_x(h) = \cos(x)$ and $f_6'(x) = \cos(x).$

- $f_7 : x \mapsto \cos(x)$

$f_7(x) = f_6\left(\frac{\pi}{2} - x\right)$ therefore

$$f_7'(x) = -f_6'\left(\frac{\pi}{2} - x\right) = -\cos\left(\frac{\pi}{2} - x\right) = -\sin(x) = f_7'(x)$$

Differentiation rules（微分法）

- Let $\alpha \in \mathbb{R}$, then $\frac{d\alpha}{dx} = 0$ （定値写像）
- **Linearity of the differentiation**: $\alpha \in \mathbb{R}$, f, g functions
$$\frac{d(\alpha f + g)}{dx} = \alpha \frac{df}{dx} + \frac{dg}{dx}$$
  - Ex: $\frac{d}{dx}(x^2 + 3x) = 2x + 3$

- **Product**: $\frac{d}{dx}(fg) = f\frac{dg}{dx} + g\frac{df}{dx}$

- **Quotient:** $\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{f\frac{dg}{dx} - g\frac{df}{dx}}{g^2}$

Ex: $\frac{d}{dx}\left(\frac{x^2+3x}{x+1}\right) = \frac{(2x+3)(x+1)-x^2-3x}{(x+1)^2} = \frac{x^2+2x+3}{(x+1)^2}$

$\frac{d}{dx}\tan(x) =$

Derivative formulas

Function （関数）	Derivative (導関数）
e^x	e^x
$\ln x$	$\frac{1}{x}$
$x^n, n \in \mathbb{Z}_{>0}$	$nx^{n-1}, n \in \mathbb{Z}_{>0}$
$\frac{1}{x^n}, \quad n \in \mathbb{Z}_{>0}$	$\frac{-n}{x^{n+1}}, \quad n \in \mathbb{Z}_{>0}$
$\cos(x)$	$-\sin(x)$
$\sin(x)$	$\cos(x)$
$\tan(x)$	$1 + \tan(x)^2 = \frac{1}{\cos(x)^2}$

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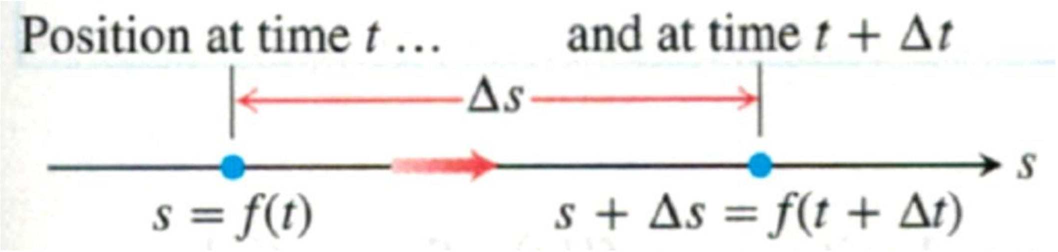
   (運動。連鎖~~法~~律、変化率）

Motion and speed I（運動と速度）



t: time
$s = f(t)$: position function

Definition: Velocity $v(t)$ is the derivative function of the position function $s = f(t)$ with respect to time.

$$v(t) = \frac{ds}{dt} = \lim_{\Delta t \to 0} \frac{f(t+\Delta t) - f(t)}{\Delta t}$$

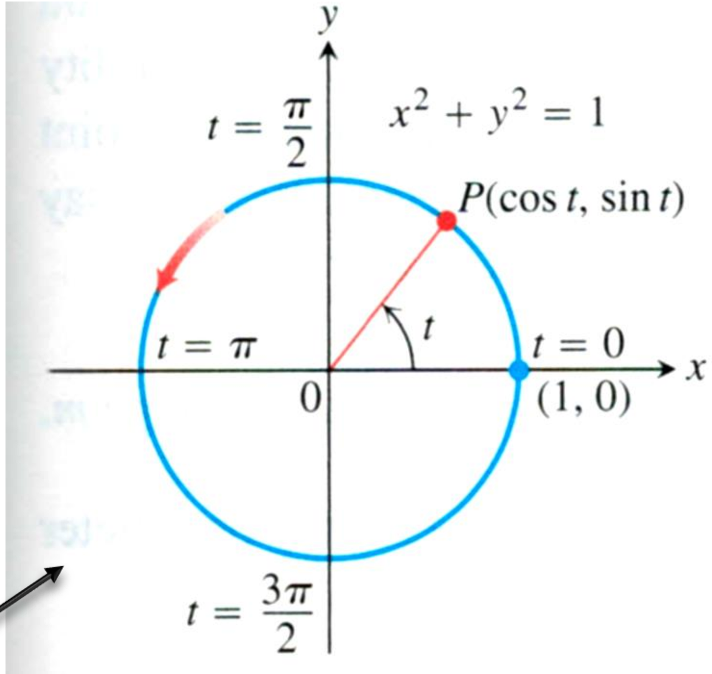
Speed $= |v(t)|$

Definition: Acceleration $a(t)$ is the derivative function of the velocity function $v = f'(t)$ with respect to time.

$$a(t) = \frac{dv}{dt} = \frac{d^2 s}{dt^2}$$

Motion and speed II

- If $\vec{s} = (x(t), y(t))$ is the position function of a body in the plane (x, y), then:
  - The plot $\{(x(t), y(t)), t \geq 0\}$ is not the graph of a function in general.
  - Ex: $x(t) = \cos(t)\,,\quad y(t) = \sin(t)$



The circle is not the graph of a function $y = f(x)$

Definition: The velocity of a body in motion in the plane is the function: $\vec{v}(t) = \left(\frac{dx}{dt}, \frac{dy}{dt}\right)$ Speed$= \| \vec{v}(t) \| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$

The acceleration is defined as : $\vec{a}(t) = \left(\frac{d^2x}{dt^2}, \frac{d^2y}{dt^2}\right)$

<u>Exercise:</u> Draw the ~~speed~~ velocity and acceleration vectors at instant $t = 1$ of a body moving around circle on the top-right graph ?

Chain rule（連鎖法律）

Theorem: If $f: u \mapsto f(u)$ is differentiable at the point $u = g(x)$ and if $g(x)$ is differentiable at x, then the composite function $(f \circ g)(x) = f(g(x))$ is differentiable at x, and :
もし $f: u \mapsto f(u)$ は$u = g(x)$において微分可能、$g(x)$はxにおいて微分可能だったら、合成関数 $(f \circ g)(x) = f(g(x))$はx における微分可能で、

$$(f \circ g)'(x) = f'\big(g(x)\big) \cdot g'(x)$$

(if $y = f(u)$ and $u = g(x)$ then $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$)

- Ex: "power chain rule" $g(x)$ is differentiable at x_0.
  $f(x) = x^n$, $(f \circ g)(x) = g(x)^n$
  $$(f \circ g)'(x) = \underbrace{n g(x)^{n-1}}_{= f'(g(x))} \cdot g'(x)$$
- Ex: Let f, g, h be there differentiable functions.
  Compute the derivative of $(f \circ g \circ h)(t) = f\Big(g\big(h(t)\big)\Big)$
  $(f \circ g \circ h)'(t) = (f \circ g)'\big(h(t)\big) \cdot h'(t) = f'\Big(g\big(h(t)\big)\Big) \cdot g'\big(h(t)\big) \cdot h'(t)$

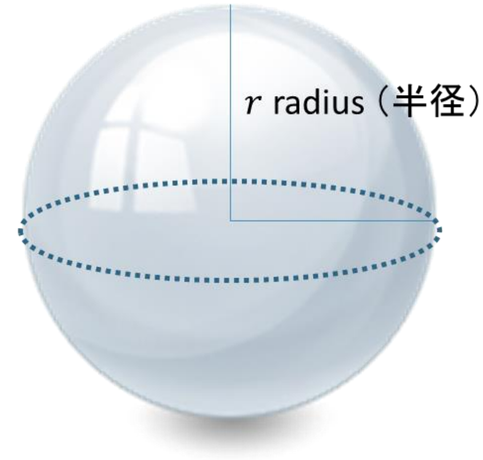
Differentiating with the chain rule

- Let $u: x \mapsto u(x)$ be a differentiable function

Function	Derivative
$e^{u(x)}$	$u'(x)e^{u(x)}$
$\ln u(x)$	$\dfrac{u'(x)}{u(x)}$
$u(x)^n, n \in \mathbb{Z}_{>0}$	$nu'(x)u(x)^{n-1}, n \in \mathbb{Z}_{>0}$
$\dfrac{1}{u(x)^n}, \quad n \in \mathbb{Z}_{>0}$	$\dfrac{-nu'(x)}{u(x)^{n+1}}, \quad n \in \mathbb{Z}_{>0}$
$\cos(u(x))$	$-u'(x)\sin(u(x))$
$\sin(u(x))$	$u'(x)\cos(u(x))$
$\tan(u(x))$	$u'(x)\left(1 + \tan\left(u(x)\right)^2\right) = \dfrac{u'(x)}{\cos(u(x))^2}$

Related rates（関係の有る変化率）

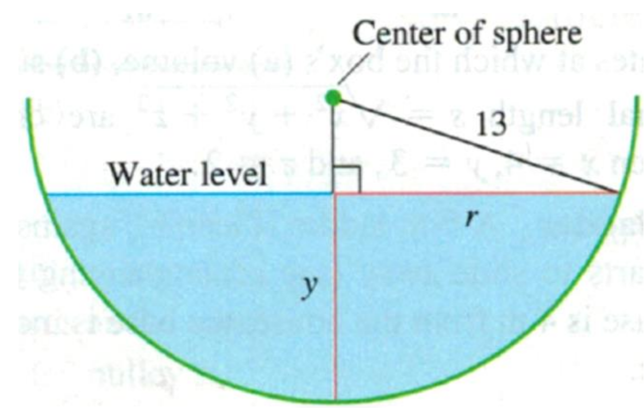
- $V = \frac{4}{3}\pi r^3$



- If the radius r varies（変分）with time t how will the volume V with respect to this rate of variation ?

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

- <u>Exercise</u>: The volume of the water is given by $V = (\pi/3)y^2(3R - y), R = 13.$
    1. At what rate is the water level y changing at $y = 8m$?
    2. Compute the radius r in function of y
    3. At what rate is the radius changing at $y = 8m$?



$$\frac{dV}{dt} = 6m^3/min$$

Homework: hand in next time pliz

1. Compute the derivative function of f
$$f(x) = \tan(\sin(x)^3)^2$$

2. In the cone cylinder , find how fast is changing the level of the water y when the water is flushed at $9\text{m}^3/\text{min}$?