Essential Mathematics for Global Leaders I

Lecture 3

Infinitely small & large: Limits

2015 April 27th (May 11th)

Xavier DAHAN
Ochanomizu Leading Promotion Center

Office:理学部2号館503
mail: dahan.xavier@ocha.ac.jp

Plan (tentative)

[4/13] L1 : introduction. Review of high-school mathematics in English.

[4/20-27] L2-3 : Functions and graphs. Plotting with Mathematica I (グラフをプロットする)

[5/7] L4 : Infinitely small and large : limits (極限)

[5/11] L5 : Continuity and differentiation (連続性と微分法)

[5/18] L6 : Differentiation II : extrema, related rates ...（極値と...）

[5/25] L7 : Differentiation III : Newton's method, Taylor's expansion
(ニュートン法とテイラー展開）

[6/1] L8 : Mid-term test. Integration I : definition, fundamental theorem of calculus　積分I.

[6/8] L9 : computation of indefinite integrals　不定積分

[6/15] L10 : Application of Integration I : length, volume and surfaces
積分の応用:長さ、面積、体積

[6/22] L11 : Application of Integration II : average, center of mass　（質量中心）, work of a force.

[6/29] L12 : Ordinary Differential equations (one variable)　常微分方程式

[7/6] L13 : Linear Differential Equations of order 2 : harmonic oscillators (small-angle pendulum, spring). 二階線形微分方程式:調和振動子　(振幅が小さい振り子、ばね)

[7/13] L14 : Ordinary Differential Equations with Mathematica. Mathematicaを利用して常微分

Program

1. Definitions （定義）
2. One-sided limits（片側極限）
3. Indeterminate form (不定形の極限）
4. Squeeze (Sandwich) theorem and applications:
   Limit of $\sin\theta/\theta$ at 0 limit of $\left(e^{h}-1\right)/h$ at 0
   （はさみうちの定理とその実用）
5. Continuity (連続性）

Reactivation of intuition (直感の再生)

What is the domain of definition of $\frac{x^2+1}{x-1}$?

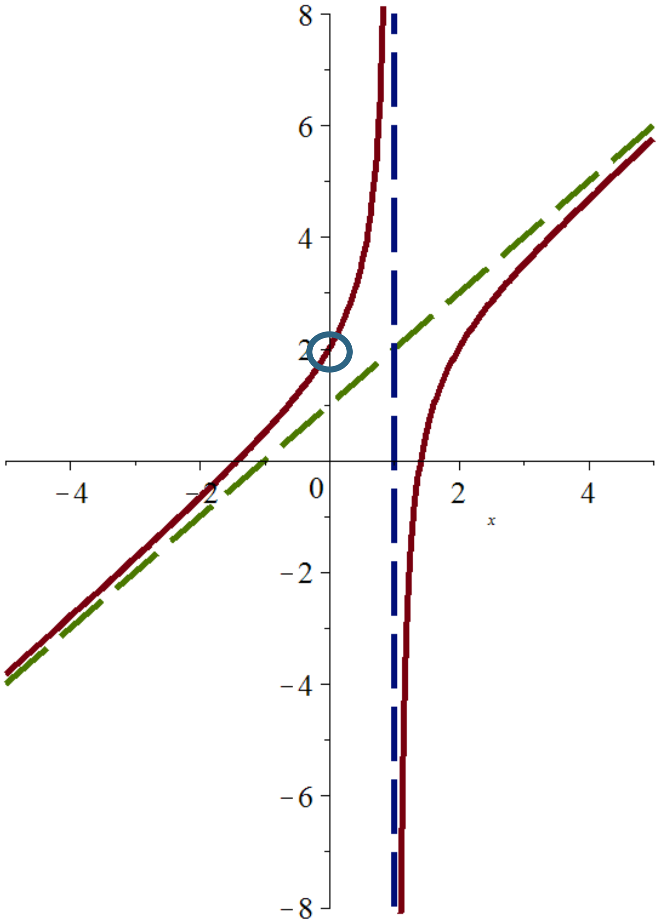
1. $\lim_{x\to 0}\frac{x^2-2}{x-1}=$

2. $\lim_{x\to 1^+}\frac{x^2-2}{x-1}=$

3. $\lim_{x\to 1^-}\frac{x^2-2}{x-1}=$

4. $\lim_{x\to \infty}\frac{x^2-2}{x-1}=$

5. $\lim_{x\to -\infty}\frac{x^2-2}{x-1}=$



区間 [−5, 5] 上の有理関数 $f(x)=\frac{x^2-2}{x-1}$ と（あれば）漸近線のグラフ。

Limit of a function (I) (関数の極限)
Beyond intuition (直感を超える)

1. Finite limit at a finite point c (cにおける有限の極限値)

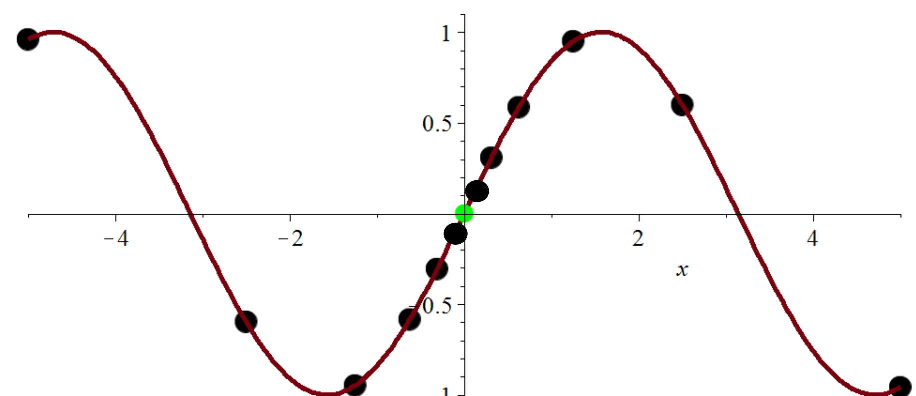
$$\lim_{x \to c} f(x) = L$$

(Reading "the limit of f as x approaches c is L"
x を c に近づけたときの $f(x)$ の極限は L である)

means that the quantity $|f(x) - L|$ can be arbitrarily small (as small as we want) if $|x - c|$ is small enough.
$|x - c|$は十分に小さければ、量$|f(x) - L|$を望む限りいくらでも小さくなることができる。

x の値を c に十分に近づければ $f(x)$の値を L に望む限りいくらでも近づけることができる。

Ex: $\lim_{x \to 0} \sin(x) =$



$x = 0$ における $f(x) = \sin(x)$ の極限値は $f(x)$ が $x = 0$ 近傍でとる値に依存

Limit of a function (II) (関数の極限)

2. Finite limit at the infinity (無限遠点における有限の極限値)

$$\lim_{x\to\infty} f(x) = L$$

(Reading "the limit of f as x approaches infinity is L")

xを無限大に近づくときの$f(x)$の極限はLである。

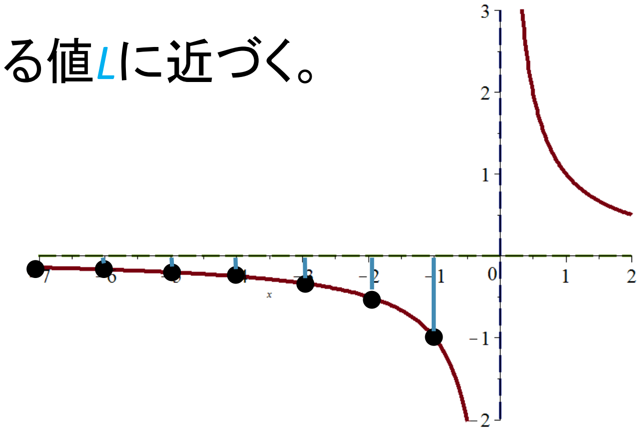
"f converges to L when x becomes arbitrarily large"

xが限りなく大きくなるとき$f(x)$はLに収束する。

means that the the quantity $|f(x) - L|$ can be arbitrarily small if x is large enough.

xが限りなく大きくなると関数$f(x)$の値がある値Lに近づく。

Ex: $\lim_{x\to-\infty} \frac{1}{x} =$



区間 [−7, 2] 上の有理関数 $f(x) = \frac{1}{x}$ と（あれば）漸近線のグラフ。

Limit of a function (III) (関数の極限)

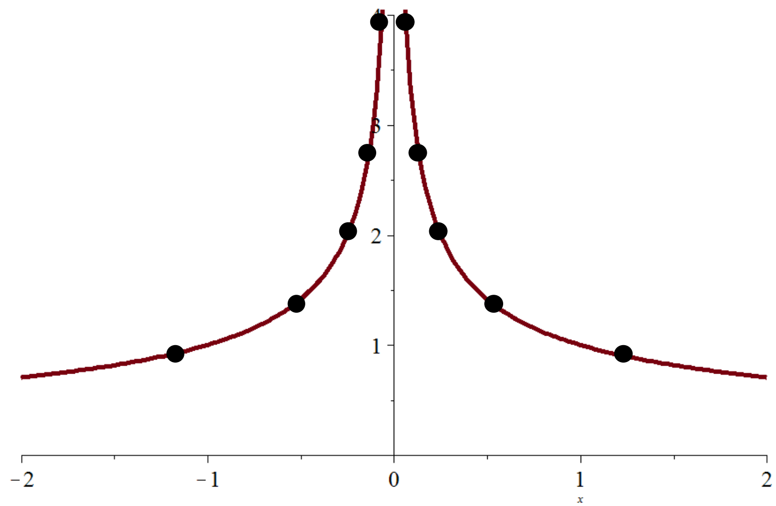
3. Infinite limit at a finite point c (cにおける無限の極限)

$$\lim_{x \to c} f(x) = \infty$$

(Reading: "The limit of f as x approaches c diverges infinity"
xがcに限りなく近づくとき関数$f(x)$は正の無限大に発散する)

means that $f(x)$ can be arbitrarily large if $|x - c|$ is small enough.
x の値を c に十分に近づければ (or $|x - c|$は十分に小さければ)
$f(x)$の値を 無限大 に望む限りいくらでも大きくなることができる。

Ex: $\lim_{x \to 0} \frac{1}{\sqrt{|x|}} =$



$x = 0$ における $f(x) = \frac{1}{\sqrt{|x|}}$ の極限値は $f(x)$ が $x = 0$ 近傍でとる値に依存

Limit of a function (IV) (関数の極限)

4. Infinite limit at the infinity (無限遠点における無限の極限)

$$\lim_{x\to\infty} f(x) = \infty$$

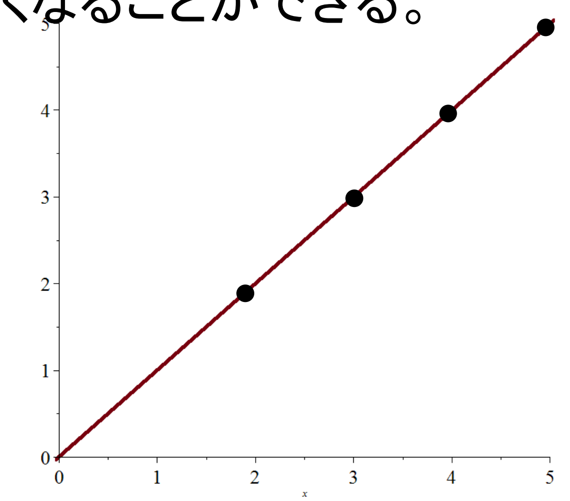
(Reading: "The limit of f as x approaches infinity diverges to infinity"
xを無限大に限りなく近づくとき関数$f(x)$は正の無限大に発散する)

means that $f(x)$ can be arbitrarily large (as large as one wants) if x is large enough.

x を無限大に十分に近づければ（or xが十分に大きければ）

$f(x)$の値を 無限大 に望む限りいくらでも大きくなることができる。

Ex: $\lim_{x\to\infty} x =$

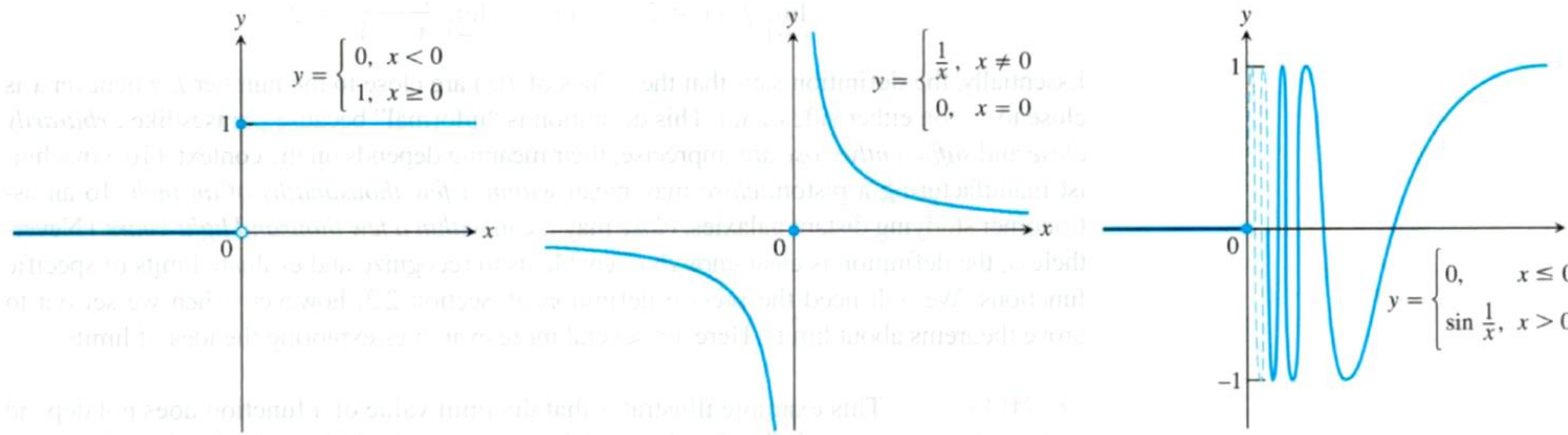


Limit of a function (V) (関数の極限)

5. There is no limit:

(or "the function f has no limit at c (or at $\pm\infty$)")

<u>Examples:</u>



(a) Unit step function $U(x)$ (b) $g(x)$ (c) $f(x)$

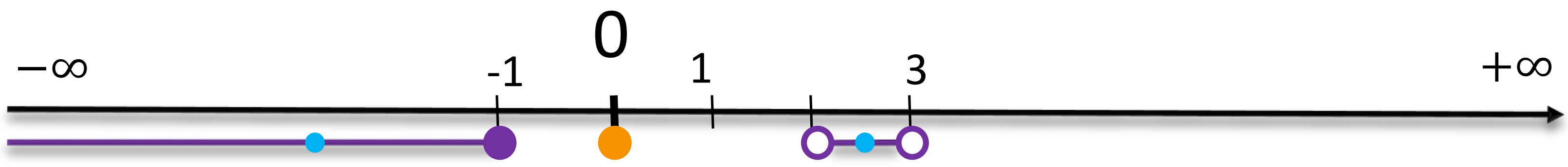
Program

1. Definitions （定義）
2. One-sided limits（片側極限）
3. Indeterminate form (不定形の極限）
4. Squeeze (Sandwich) theorem and applications:
   Limit of $\sin\theta/\theta$ at 0 limit of $\left(e^h-1\right)/h$ at 0
   （はさみうちの定理とその実用）
5. Continuity (連続性）

One-sided limit (片側極限)

- **Endpoints, interior and isolated points** (端点、内点 と 孤立点)

$f: E \subset \mathbb{R} \to \mathbb{R}$ a function with E its domain of definition:



$$E = (-\infty, -1] \cup \{0\} \cup (2,3)$$

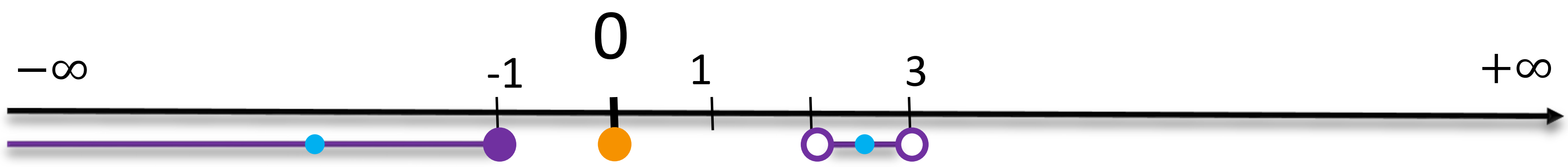
$-1,2,3$ are endpoints（端点）. Other points in E are interior points (内点 they are inside an interval). 0 is an isolated point （孤立点）.

- **Case1→ c is interior:** $\lim_{x \to c^{+} \text{ or } -} f(x) = L \text{ (or } N)$ "right (or left)-hand limit"

Reading: "…x approaches c to the right (left)…."
Definition: (in the previous slide, replace " $|x - c|$ is small enough" by "$x - c$ is small enough and $x > c$", "$c - x$ is small enough and $x < c$")

$\lim_{x \to c} f(x)$ exists ⇔ $\lim_{x \to c^{+}} f(x)$ and $\lim_{x \to c^{-}} f(x)$ exist and are equal.

One-sided limit (II)（片側極限）



$E = (-\infty, -1] \cup \{0\} \cup (2,3)$

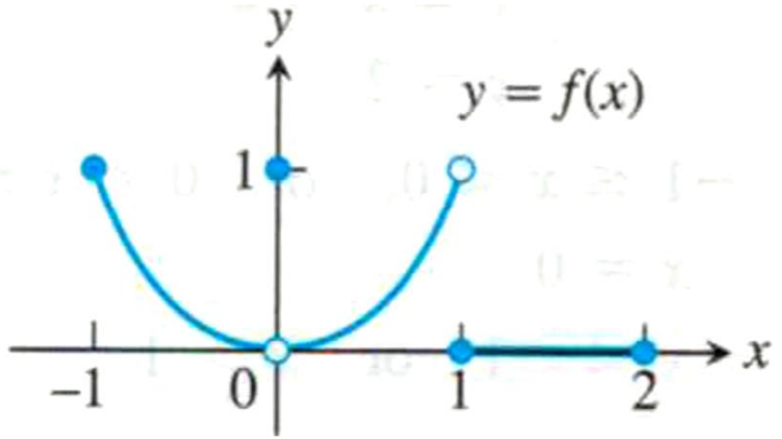
- **Case 2**→ c is endpoint (for example, c is a right endpoint like $-1,3$ above. A left endpoint is 2)
  Then we can define a left-hand limit when $x \to c^-$ but we cannot define a right-hand limit $x \to c^+$ (f is not defined at the right of c !)

$\lim_{x \to c} f(x)$ exists ⇔ $\lim_{x \to c^-} f(x)$ exists (and then they are both equal)

- **Case 3**→ c is isolated: no limit, nor right-hand neither left-hand limit.
  極限が無い。左も右も片側極限が無い。

Exercise:

True ○ or false × ?



a) $\lim_{x \to -1^+} f(x) = 1$

b) $\lim_{x \to 0^-} f(x) = 0$

c) $\lim_{x \to 0^-} f(x) = 1$

d) $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x)$

e) $\lim_{x \to 0} f(x)$ exists

f) $\lim_{x \to 0} f(x) = 0$

g) $\lim_{x \to 0} f(x) = 1$

h) $\lim_{x \to 1} f(x) = 1$

i) $\lim_{x \to 1} f(x) = 0$

j) $\lim_{x \to 2^-} f(x) = 2$

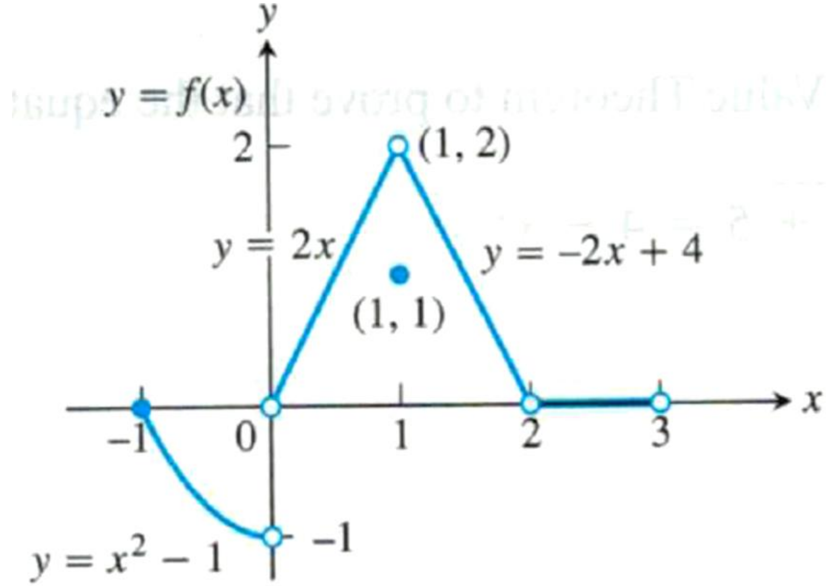
k) $\lim_{x \to 1^-} f(x)$ does not exist

l) $\lim_{x \to 2^+} f(x) = 0$

a)	b)
c)	d)
e)	f)
g)	h)
i)	j)
k)	l)

Exercise:

$$f(x) = \begin{cases} x^2 - 1, & if \quad -1 \le x < 0 \\ 2x, & if \quad 0 < x < 1 \\ 1, & if \quad x = 1, \\ -2x + 4, & if \quad 1 < x < 2 \\ 0, & if \quad 2 < x < 3 \end{cases}$$



1. For $c = -1$ and $c = 0$, answer the questions:
   a. Does $f(c)$ exists ?
   b. Does $\lim_{x \to c^+} f(x)$ exits ?
   c. What about $\lim_{x \to c^-} f(x)$?

2. Is f defined at $x = 2$?
   What is the value of $f(3)$?

Exercise:

1. Is there a (right, left) limit at 1 of the function $f: x \mapsto \frac{x^2-1}{x-1}$?

2. Same question for: $g: x \mapsto \begin{cases} \frac{x^2-1}{x-1} & \text{if } x \neq 1 \\ 1 & \text{if } x = 1 \end{cases}$

3. Same question for: $h: x \mapsto \frac{x^2+x-2}{x^2-x}$

Program

1. Definitions （定義）

2. One-sided limits（片側極限）

3. Indeterminate form (不定形の極限）

4. Squeeze (Sandwich) theorem and applications:
   Limit of $\sin\theta/\theta$ at 0   limit of $(e^h-1)/h$ at 0
   （はさみうちの定理とその実用）

5. Continuity (連続性）

Operations on limits and indeterminate forms (極限の演算と不定形)

$f: E \subset \mathbb{R} \to \mathbb{R}$, $g: F \subset \mathbb{R} \to \mathbb{R}$ be two functions and let $c \in E \cap F$

$$\lim_{x \to c} f(x) = L \qquad \lim_{x \to c} g(x) = M$$

$f+g$	$f-g$	$f \cdot g$	f/g	$f^n, n \in \mathbb{Z}_{>0}$	$\sqrt[n]{f} = f^{\frac{1}{n}}, n \in \mathbb{Z}_{>0}$	$f^{\frac{n}{m}}$
$L+M$	$L-M$	$L \cdot M$	L/M , $M \neq 0$	L^n	$\sqrt[n]{L}$	$L^{\frac{n}{m}}$

Value of a limit at a point c or at ∞

f	g	f/g	$f \times g$	$f-g$
0	0	?	0	0
∞	± 0	$\pm\infty$	?	∞
± 0	∞	0	?	$-\infty$
∞	∞	?	∞	?

Exercise:

1. What is the domain of definition (定義域）of the functions:
   a. $f(x) = \frac{1+x+\sin(x)}{3\cos(x)}$
   b. $g(x) = \frac{x^4+x^2-1}{x^2+5}$
   c. $h(x) = \sqrt{4x^2 - 3}$

2. For each endpoints c of the domains, find:
   $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$

3. Tell if the limits below exist, and if so compute it.

$$\lim_{x\to\frac{\sqrt{3}}{2}^-} h(x) = \quad , \qquad \lim_{x\to\frac{\sqrt{3}}{2}^+} h(x) = \quad , \qquad \lim_{x\to\frac{\sqrt{3}}{2}} h(x) =$$

Simplification (単純化)

Simplify (単純化) the following expressions (数式）to find the limits.

1. $f(x) = \frac{\sqrt{x^2+100}-10}{x^2}$. Limit at 0 is an indeterminate form of type $\frac{0}{0}$.
2. $\lim_{x\to -1} \frac{\sqrt{x^2+8}-3}{x+1}$
3. $\lim_{x\to\infty} (\sqrt{x+9} - \sqrt{x+4})$

Behavior at ∞ of rational functions (有理関数の無限遠での振る舞い)

- Ex: $\lim_{x\to\infty}\frac{x^{10}+x+1}{x^9-1}$ is an indeterminate form of type: $\frac{\infty}{\infty}$.

- (Lesson 2 page 16) Rational function:

$$f(x)=\frac{a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0}{b_mx^m+\cdots+b_1x+b_0}=\frac{A(x)}{B(x)},\quad a_n\neq 0, b_m\neq 0$$

- **Theorem:** $\lim_{x\to\pm\infty}\frac{A(x)}{B(x)}=\lim_{x\to\pm\infty}\frac{a_nx^n}{b_mx^m}\begin{cases}=\frac{a_n}{b_m} & \text{if } n=m\\ 0 & \text{if } n<m\\ \pm\infty & \text{if } n>m\end{cases}$

It behaves like $\frac{a_nx^n}{b_mx^m}$ at ∞。

(無限遠で$\frac{a_nx^n}{b_mx^m}$のように振る舞う)

(+ or – depends on the signs of a_n and b_m) (a_nとb_mの符号に依存する)

Proof (証明)

- $A(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = a_n x^n \left(1 + \frac{a_{n-1}}{a_n}\frac{1}{x} + \cdots + \frac{a_1}{a_n}\frac{1}{x^{n-1}} + \frac{a_0}{a_n}\frac{1}{x^n}\right)$
- $B(x) = b_m x^m + \cdots + b_1 x + b_0 = b_m x^m \left(1 + \frac{b_{m-1}}{b_m}\frac{1}{x} + \cdots + \frac{b_1}{b_m}\frac{1}{x^{m-1}} + \frac{b_0}{b_m}\frac{1}{x^m}\right)$

$\lim\limits_{x\to\pm\infty} \frac{1}{x^i} = 0$ for $i > 0 \Rightarrow$ terms in blue go to 0 when $x \to \infty$

Therefore

- $\lim\limits_{x\to\pm\infty} A(x) = \lim\limits_{x\to\pm\infty} a_n x^n (1 + 0)$ and $\lim\limits_{x\to\pm\infty} B(x) = \lim\limits_{x\to\pm\infty} b_m x^m (1 + 0)$
- Thus, $\lim\limits_{x\to\pm\infty} \frac{A(x)}{B(x)} = \lim\limits_{x\to\pm\infty} \frac{a_n x^n}{b_m x^m}$

Exercise

Use the theorem to compute the following limits of rational functions.

1. $\lim_{x \to \infty} \frac{-x^2+1}{3x^3+2x^2+1}$

2. $\lim_{x \to \infty} \frac{-x^3+1}{3x^3+2x^2+1}$

3. $\lim_{x \to 1} \frac{\frac{3}{(x-1)^2}+\frac{1}{(x-1)}-1}{\frac{1}{(x-1)^2}+1}$

(think of changing of variable 変数の変換)

4. Use the same trick as in the proof to compute:

$$\lim_{x \to \infty} \frac{x + \sin(x) + \sqrt{x}}{x + \sin(x)}$$

Program

1. Definitions (定義）

2. One-sided limits（片側極限）

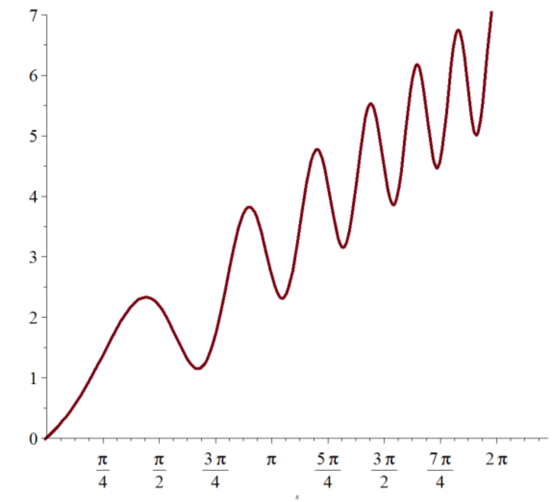
3. Indeterminate form (不定形の極限）

4. Squeeze (Sandwich) theorem and applications:
   Limit of $\sin\theta/\theta$ at 0 limit of $\left(e^h-1\right)/h$ at 0
   （はさみうちの定理とその実用）

5. Continuity (連続性）

The sandwich (or squeeze or pinching) theorem (はさみうちの定理）

- Ex: $\lim_{x\to\infty} x + \sin(x^2) =$
- $\lim_{x\to 0} x \cdot \sin(1/x) =$



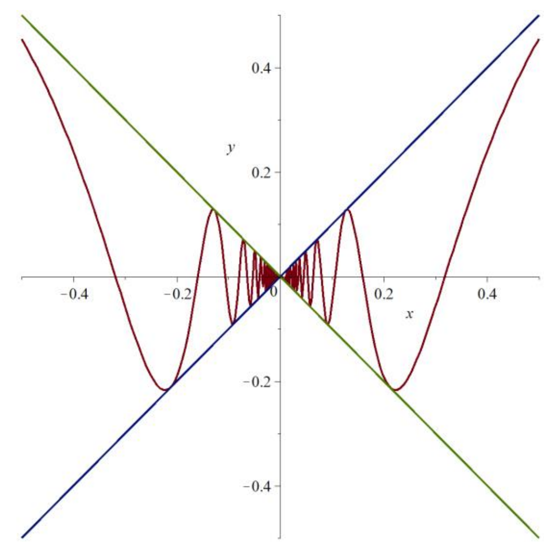
Theorem: Let f, g, h be 3 real functions such that

$$f(x) \leq g(x) \leq h(x)$$

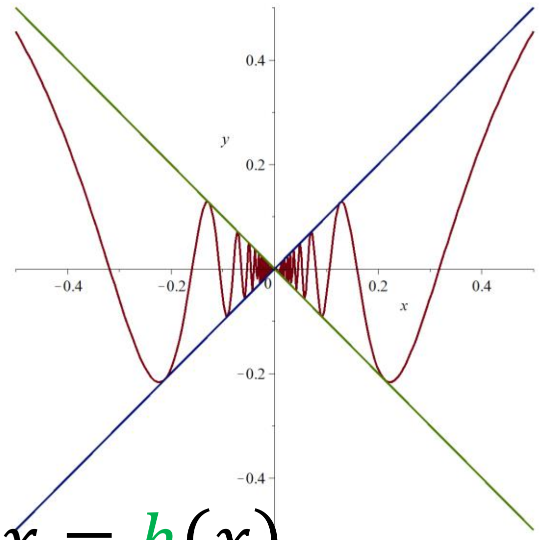
for all points x in an interval $[a, b]$, except maybe at $c \in (a, b)$. If

$$\lim_{x\to c} f(x) = \lim_{x\to c} h(x) = L$$

Then $\boxed{\lim_{x\to c} g(x) = L}$



Application of the sandwich theorem



- $\lim_{x\to 0} x \cdot \sin(1/x) =$
- $-1 \leq \sin(1/x) \leq 1\,,\ x \neq 0$
- Therefore,
$$f(x) = -x \leq x \cdot \sin(1/x) \leq x = h(x)$$
- $\lim_{x\to 0} f(x) = \lim_{x\to 0} h(x) = 0$
- By the sandwich theorem, wo obtain:
$$\lim_{x\to 0} x \cdot \sin(1/x) = 0$$

The sandwich theorem at infinity

- **Theorem:** If $f(x) \leq g(x)$ for all x in some interval $I = [a, b]$ (or $I = [a, \infty)$) except maybe at a point $c \in [a, b]$
- and if: $\lim_{x \to c \ or \ \infty} f(x)$ and $\lim_{x \to c \ or \ \infty} g(x)$ both exist, then

$$\lim_{x \to c} f(x) \leq \lim_{x \to c} g(x) \quad \left(\text{or } \lim_{x \to \infty} f(x) \leq \lim_{x \to \infty} g(x) \right)$$

Corollary (系, = consequence) : With the same f and g as above, if:

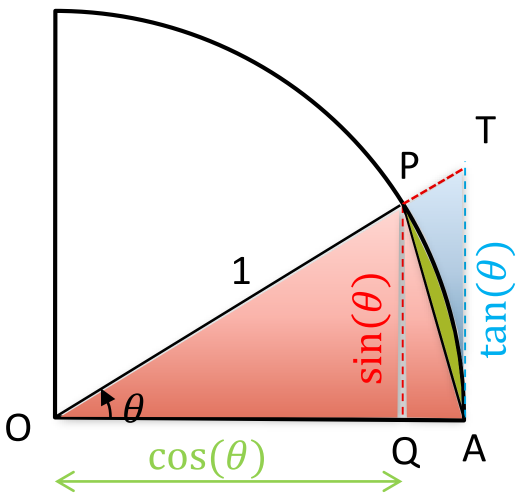
$\lim_{x \to c} f(x) = \infty$ (or $\lim_{x \to \infty} f(x) = \infty$) then

$$\lim_{x \to c} g(x) = \infty \quad (\text{or } \lim_{x \to \infty} g(x) = \infty)$$

- **Application**（実用） $\lim_{x \to \infty} x + \sin(x^2) = ??$
- $f(x) = x - 1 \leq x + \sin(x^2)$, $\lim_{x \to \infty} f(x) = \infty$, by the sandwich theorem at infinity, $\lim_{x \to \infty} x + \sin(x^2) = \infty$.

Limit of $\sin\theta/\theta$ at 0

- $\lim_{\theta\to 0}\frac{\sin\ \theta}{\theta}$ is an indeterminate form of type $\frac{?=}{?=}$



- Area (面積） of triangle（三角形）OPA: Δ_{OPA}
- Area of sector (扇形) OAP: Σ_{OAP}
- Area of triangle OAT: Δ_{OAT}
- $\Delta_{OAP} \le \Sigma_{OAP} \le \Delta_{OAT}$
- $\frac{1}{2}\sin(\theta) \le \pi r^2 \times \left(\frac{\theta}{2\pi}\right) \le \frac{1}{2}\tan(\theta)$

$\times\frac{2}{\sin(\theta)}\ (\theta > 0)$

- $1 \le \frac{\theta}{\sin(\theta)} \le \cos(\theta) \quad \Rightarrow \quad \frac{1}{\cos(\theta)} \le \frac{\sin(\theta)}{\theta} \le 1$
- Since $\lim_{\theta\to 0^+}\cos(\theta) = 1$, by the sandwich theorem:

$$\lim_{x\to 0^+}\frac{\sin(\theta)}{\theta} = 1$$

Exercise

1. Compute $\lim_{x \to 0} \frac{\sin(2x)}{5x}$
2. Compute $\lim_{h \to 0} \frac{\cos(h)-1}{h}$
3. Let $x \in \mathbb{R}$, fixed（変数でない）.
   Show that $\lim_{h \to 0} \frac{\sin(x+h)-\sin(x)}{h} = \cos(x)$
4. $\lim_{x \to 0} \frac{\sin(\sin(x))}{x}$
5. $\lim_{x \to 0} \frac{\sin(x)}{\sin(\sqrt{x})}$

Limit of $\left(e^h - 1\right)/h$ at 0

- $\lim_{h\to 0} \frac{e^h-1}{h}$ is an undetermined form of type $\frac{0}{0}$.

- (lesson 2, page 17) $x \to e^x$ was defined as the unique function:

$f(1) = e \approx 2.76$
$f(a+b) = f(a)f(b)$ (+ graph of f is smooth, 連続的)

- To compute the limit, we need to know what is $e \simeq 2.76..$

Proof next slide
（次に証明）

- **Definition**: $e = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n$ $\left(= \lim_{n\to\infty} \sum_{k=0} C_n^k \frac{1}{n^k}\right)$

We have: $\lim_{n\to\infty} n\left(e^{\frac{1}{n}} - \left(1+\frac{1}{n}\right)\right) = \lim_{n\to\infty} n\left(e^{\frac{1}{n}} - 1\right) - 1 = 0$

Thus, $\lim_{n\to\infty} n\left(e^{\frac{1}{n}} - 1\right) = 1$. Let $h = 1/n$. Then $\lim_{n\to\infty} \frac{e^{\frac{1}{n}}-1}{1/n} = \boxed{\lim_{h\to 0} \frac{e^h-1}{h} = 1}$

Proof that $\lim_{n\to\infty} n\left(e^{1/n} - (1 + 1/n)\right) = 0$

$$e - (1 + 1/n)^n$$
$$= \left(e^{1/n} - (1 + 1/n)\right)\left(e^{\frac{n-1}{n}}(1 + 1/n) + e^{\frac{n-2}{n}}(1 + 1/n)^2 + \cdots \right.$$
$$\left. + e^{\frac{2}{n}}(1 + 1/n)^{n-1}\right)$$
$$= \left(e^{1/n} - (1 + 1/n)\right) B_n$$

Each term $e^{k/n}(1 + 1/n)^{n-k} > 1$. Therefore, $B_n > n - 1$.

By assumption,

$$\lim_{n\to\infty} \left(e^{1/n} - (1 + 1/n)\right) B_n = \lim_{n\to\infty} e - (1 + 1/n)^n = 0$$

Therefore, since $B_n > n - 1$.

$$\lim_{n\to\infty} n\left(e^{1/n} - (1 + 1/n)\right) = 0.$$

Program

1. Definitions （定義）
2. One-sided limits（片側極限）
3. Indeterminate form (不定形の極限）
4. Squeeze (Sandwich) theorem and applications:
   Limit of $\sin\theta/\theta$ at 0 limit of $\left(e^h - 1\right)/h$ at 0
   （はさみうちの定理とその実用）
5. Continuity (連続性）

Continuity at a point and on an interval
点と区間で連続性

Definition: $f: E \subset \mathbb{R} \to \mathbb{R}$ is continuous at $\boldsymbol{c} \in \boldsymbol{E}$ if:

Case1: c is an interior point of E

1) $\lim_{x \to c} f(x)$ exists and is not $\pm\infty$
2) $\lim_{x \to c} f(x) = f(c)$

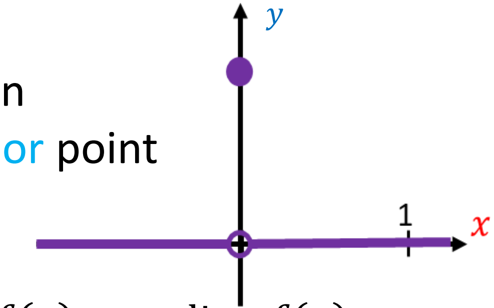
Case 2: c is an endpoint and $c \in E$ (say here a left endpoint)

1) $\lim_{x \to c^+} f(x)$ exists
2) $\lim_{x \to c^+} f(x) = f(c)$

Definition: f is continuous on an interval I if it continuous at every point of I.

Intuitive meaning: it is possible to draw the graph of f on I without lifting a pen.

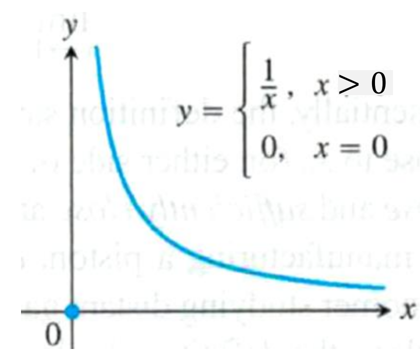
$$f(x) = \begin{cases} 0 & if \quad x \neq 0 \\ 1 & if \quad x = 0 \end{cases}$$



0 is an interior point

$\lim_{x \to 0^-} f(x) =$ $\lim_{x \to 0^+} f(x) =$

Function defined over $\mathbb{R}$, continuous on $\mathbb{R}^*$



0 is and left endpoint

$\lim_{x \to 0^+} f(x) =$

Function defined over $\mathbb{R}_{\geq 0}$, continuous on $\mathbb{R}_{>0}$ 32

Which functions are continuous ?

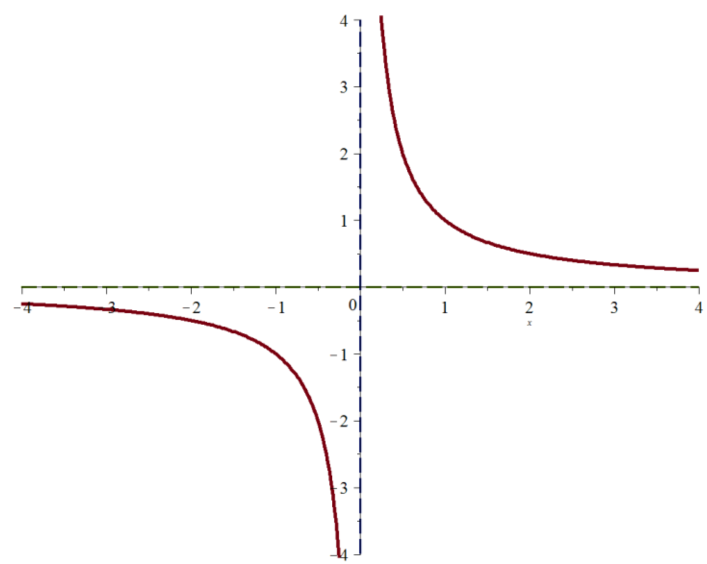
We have use this property all the time to compute limits !
極限を計算した時に、連続性を頻繁に使った。

(No worry ☺) Theorem 1: All the functions studied in lesson 2 (polynomials, n-th root, rational function, exp, log, sin, cos, tan) are continuous on their domain of definition

- **Remark:** $f: \mathbb{R}^\star \to \mathbb{R},\ x \mapsto \frac{1}{x}$ continuous on its domain of definition $\mathbb{R}^\star$. But not continuous at 0)

Theorem 2 (composition 合成): If f is continuous at a point c and g continuous at $f(c)$, then $g \circ f$ is continuous at c:

$$\lim_{x \to c} g \circ f\,(x) = g(\lim_{x \to c} f(x)) = g(f(c))$$



区間 [−4, 4] 上の有理関数 $f(x) = \frac{1}{x}$ と（あれば）漸近線のグラフ。

Continuity of the inverse

Theorem (Inverse　逆関数): (lesson 2, page 19) The inverse f^{-1} of a continuous function f, is continuous on its domain of definition.

- **Example:** ln is the inverse function of $x \to e^x$ which is continuous (by definition). Therefore, ln is continuous on $\mathbb{R}_{>0}$.
- Ex: $\lim_{h \to 0} \frac{\ln(1+h)-\ln(1)}{h} \quad = \lim_{h \to 0} \ln(1+h)^{\frac{1}{h}}$

Since $x \to \ln(x)$ in continuous at 1, this is equal to:

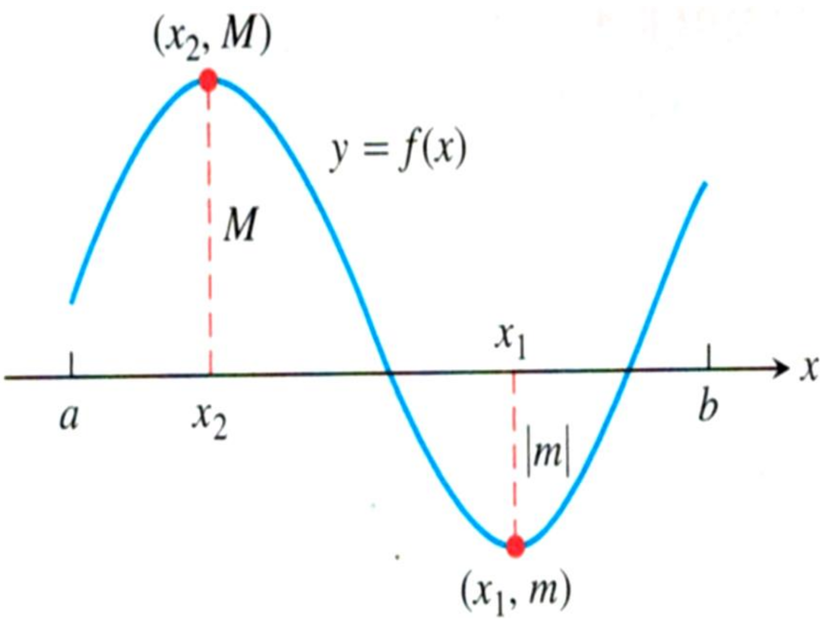
$$\ln\left(\lim_{h \to 0}(1+h)^{\frac{1}{h}}\right) = \ln(e) = 1.$$

Intermediate & extreme value theorems (中間値の定理、最大値最小値定理)

Theorem: Let $f: [a, b] \to \mathbb{R}$ be a continuous function.
Let $m := \min_{a \le x \le b} f(x)$ and $M := \max_{a \le x \le b} f(x)$.
For any $c \in [m, M]$, there exists $x_c \in [a, b]$ such that: $f(x_c) = c$.

Remark:

- This is equivalent to $f([a, b]) = [m, M]$
- Close interval 閉区間 $[a, b]$ is important.
  This is not true in general if $f: (a, b) \to \mathbb{R}$
- Very intuitive theorem, but the proof is not.(直観的な定理であるが、その証明は直観ではない)。



Homework (Hand in on May 7th pliz)

1. $\lim \dfrac{x^2-4x+4}{x^3+5x^2-14x}$ as $x \to 0^+$ and as $x \to 2^+$
2. $\lim\limits_{x\to 1} \dfrac{1-\sqrt{x}}{1-x}$
3. $\lim\limits_{x\to \cancel{1}\ 0^+} \dfrac{\frac{1}{2+x}-\frac{1}{2}}{x}$
4. ~~$\lim\limits_{x\to \pi} \sin\left(\dfrac{x}{2}+\sin(x)\right)$~~
5. $\lim\limits_{x\to -\infty} \dfrac{2x^2+3}{5x^2+7}$
6. $\lim\limits_{x\to -\infty} \dfrac{x^4+x^3}{12x^3+128}$

Homework (II)

7. $\lim_{x\to-\infty}\left(\frac{x^2+x-1}{8x^3-3}\right)^{1/3}$

8. (a) Show that: $\lim_{x\to 0}\frac{\sqrt{1+x}-1}{x}=\frac{1}{2}$.

   (b) Deduce that: $\lim_{x\to\infty}\sqrt{x^2+x}-x=\frac{1}{2}$

   (c) compute $\lim_{x\to\infty}(2x - \sqrt{4x^2+3x-2})$ *(*)*

9. $\lim_{\theta\to\infty\ 0^+}\frac{\cos(\theta)-1}{\theta}$

10. Let $x_o \in \mathbb{R}_{>0}$ fixed. Show that:

$$\lim_{h\to 0}\frac{\ln(x_0+h)-\ln(x_0)}{h}=\frac{1}{x_0}$$