Essential Mathematics for Global Leaders I

Lecture 2

Functions and graphs

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Plan (tentative)

[4/13] L1 : introduction. Review of high-school mathematics in English.

[4/20-27] L2-3 : Functions and graphs. Plotting with Mathematica I (グラフをプロットする)

[5/7] L4 : Infinitely small and large : limits (極限)

[5/11] L5 : Continuity and differentiation (連続性と微分法)

[5/18] L6 : Differentiation II : extrema, related rates ...（極値と...）

[5/25] L7 : Differentiation III : Newton's method, Taylor's expansion (ニュートン法とテイラー展開）

[6/1] L8 : Mid-term test. Integration I : definition, fundamental theorem of calculus　積分I.

[6/8] L9 : computation of indefinite integrals　不定積分

[6/15] L10 : Application of Integration I : length, volume and surfaces
積分の応用:長さ、面積、体積

[6/22] L11 : Application of Integration II : average, center of mass （質量中心）, work of a force.

[6/29] L12 : Ordinary Differential equations (one variable)　常微分方程式

[7/6] L13 : Linear Differential Equations of order 2 : harmonic oscillators (small-angle pendulum, spring). 二階線形微分方程式:調和振動子 (振幅が小さい振り子、ばね)

[7/13] L14 : Ordinary Differential Equations with Mathematica. Mathematicaを利用して常微分

Today's program

1. Basics （基礎）

2. Powers, Polynomials, Rational functions
（ベキ乗、多項式、有理関数）

3. Exponential & Logarithm functions
（指数と対数関数）

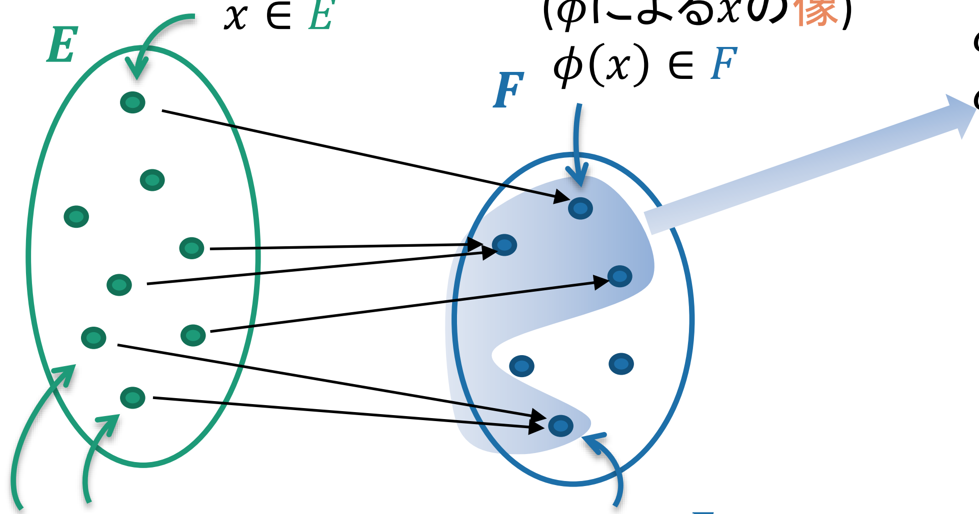
4. Trigonometric functions （三角関数）

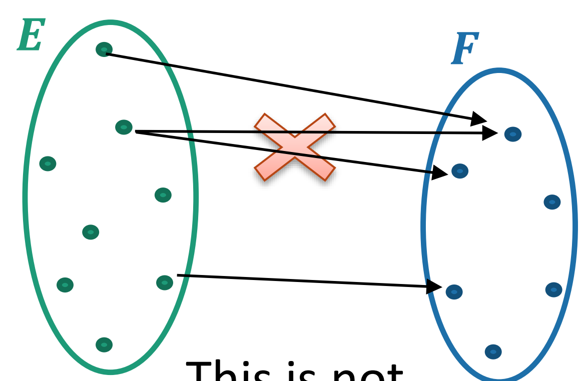
Functions and Maps (関数と写像)

Map (or a function)ϕ from the set E to the set F

(集合Eから集合Fへの写像(または関数)ϕ)

E is the domain of ϕ
Eはfの定義域



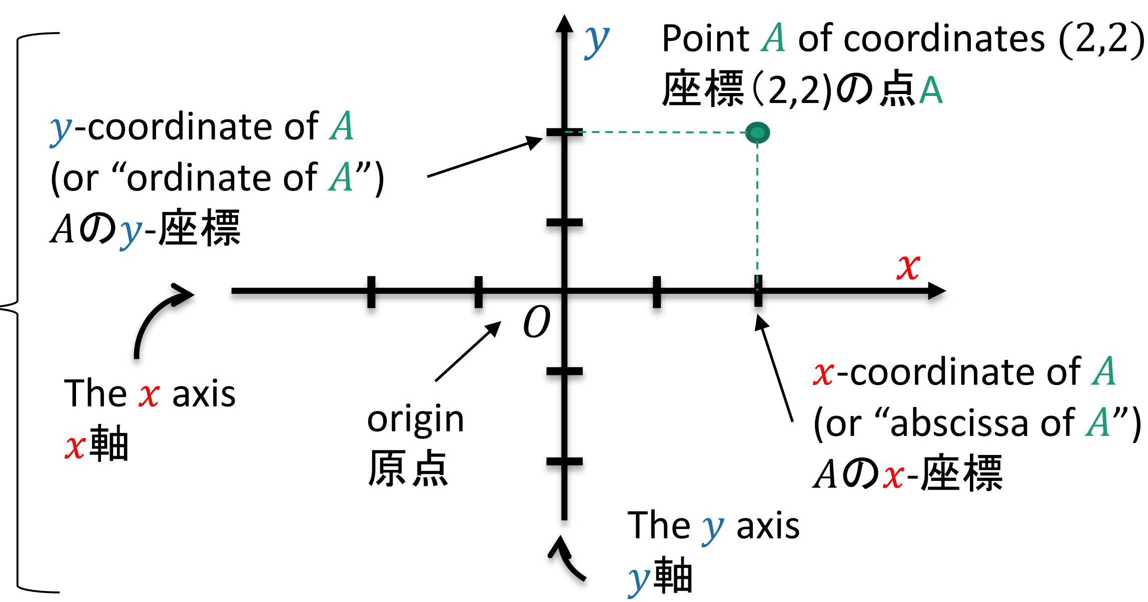


Real valued functions (実数値関数)

- $f: E \to \mathbb{R}$ is called a real valued function.
  (Reading: "f from E to R")
- $f: E \subset \mathbb{R} \to \mathbb{R}$ is a function of a real variable （実関数）

Definition: The graph of f is the set of points $\{(x, f(x)) : x \in E\}$. A visual representation is called a plot or plotting. (その視覚的な表示はプロットという)。

Frame of the xy plane
xy表面上の座標系

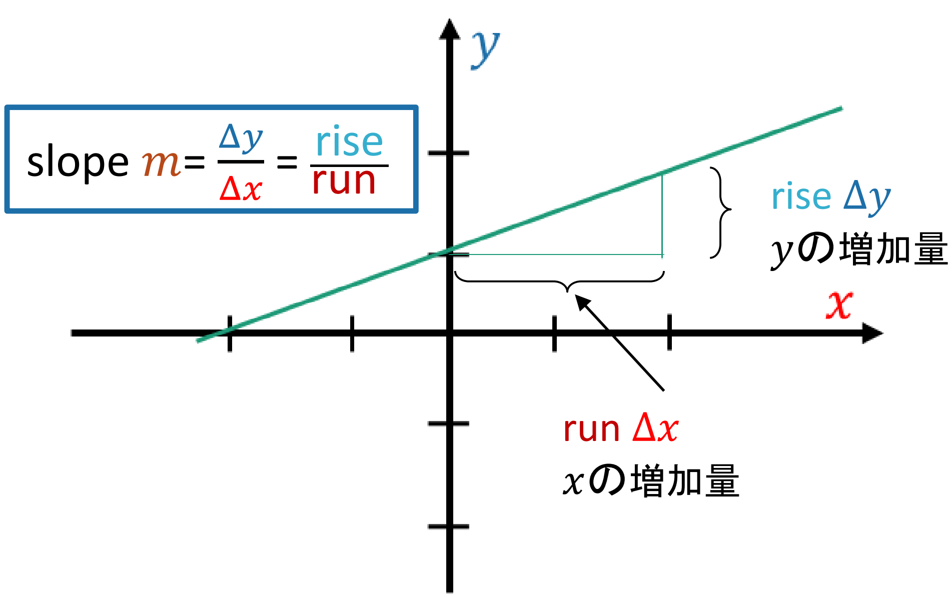


The straight line （直線）

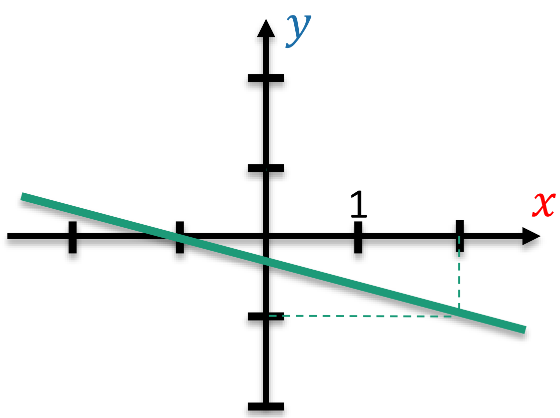
- **<u>Equation:</u>** $f: x \mapsto mx + b,$

(<u>Reading:</u> “f the function that maps x to $mx + b$”)

($m, b \in \mathbb{R}$ constants（定数）, x variable （変数）)

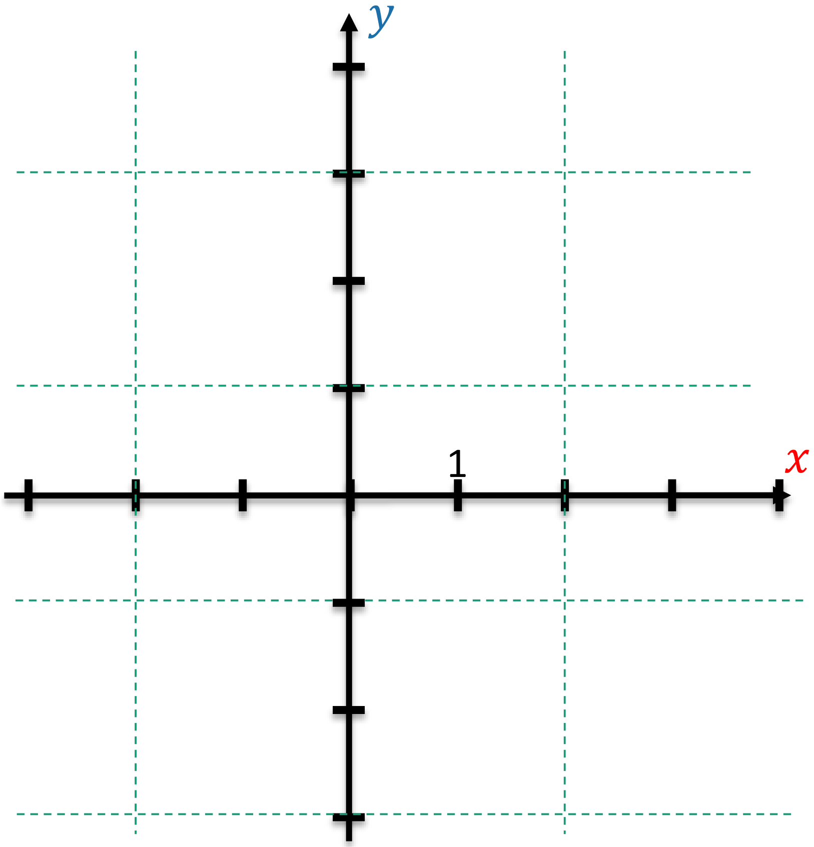


- m is called the slope (傾き)
- What is the slope of the line below ?



Exercise:

- Draw the graph of the function $f: x \mapsto |2x - 2|$



Today’s program

1. Basics （前菜）
2. Powers, Polynomials, Rational functions（ベキ、多項式、有理関数）
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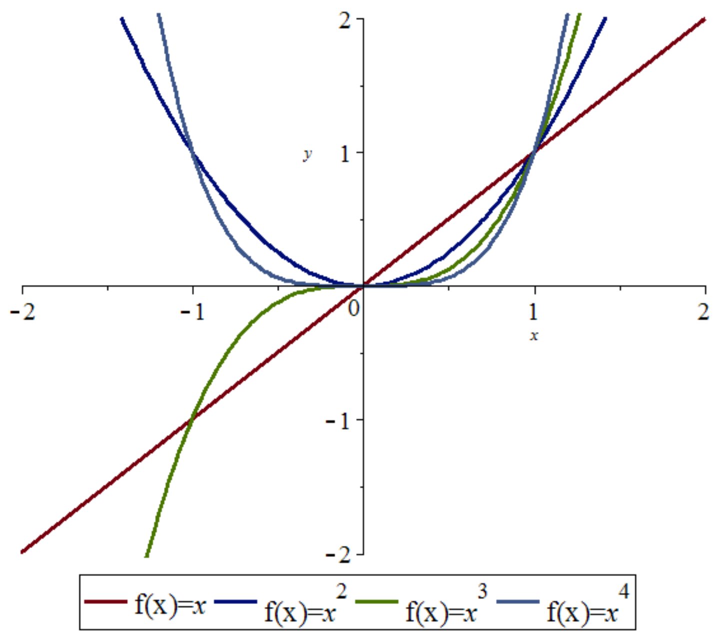
4. Trigonometric functions （三角関数）

Power functions（ベキ関数）

- Equation: $f: x \mapsto x^{a}$, a is a constant.
(Reading: "function f that maps x to x to the (power of) a")

Case 1: $a \in \mathbb{Z}_{>0}$ (Reading: a is a positive integer 正の整数)
The domain of definition（定義域） of f is $\mathbb{R}$



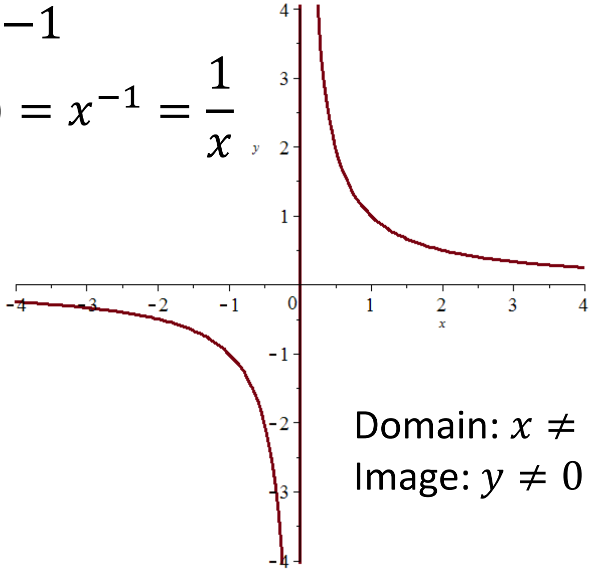
Power functions (II) (ベキ関数) $x \mapsto x^a$

<u>Case 2:</u> $a \in \mathbb{Z}_{<0}$ (<u>Reading:</u> a is a negative integer 負の整数)
The domain of definition（定義域） of f is

$$\mathbb{R}^* = (-\infty, 0) \cup (0, \infty)$$

$a = -1$

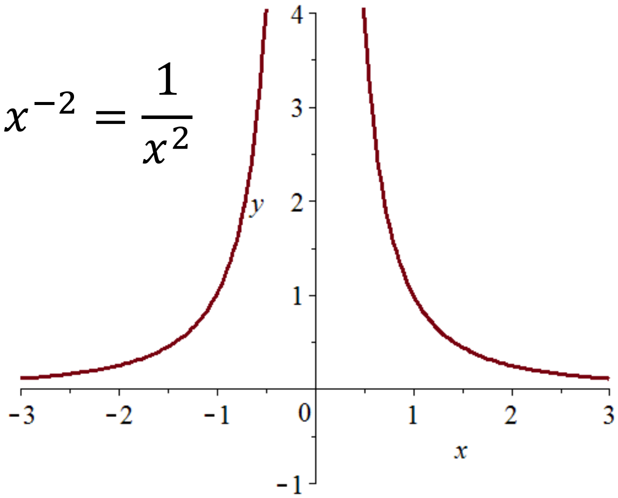
$$f(x) = x^{-1} = \frac{1}{x}$$



Domain: $x \neq 0$
Image: $y \neq 0$

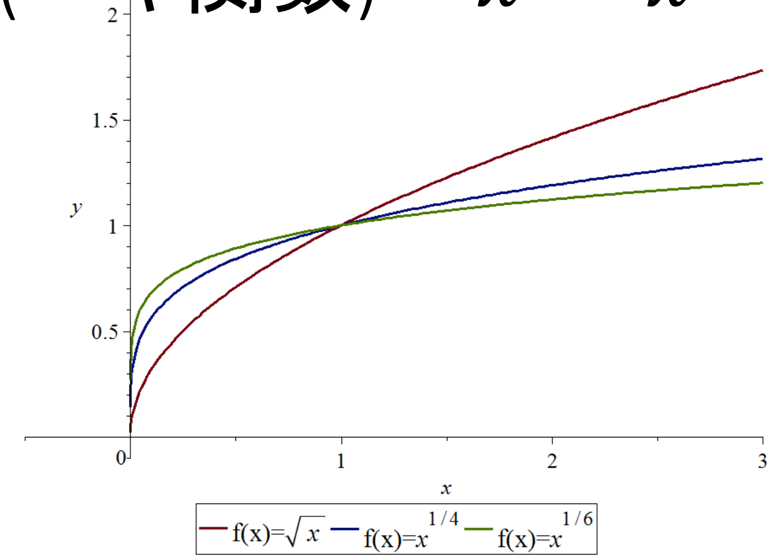
$a = -2$

$$f(x) = x^{-2} = \frac{1}{x^2}$$

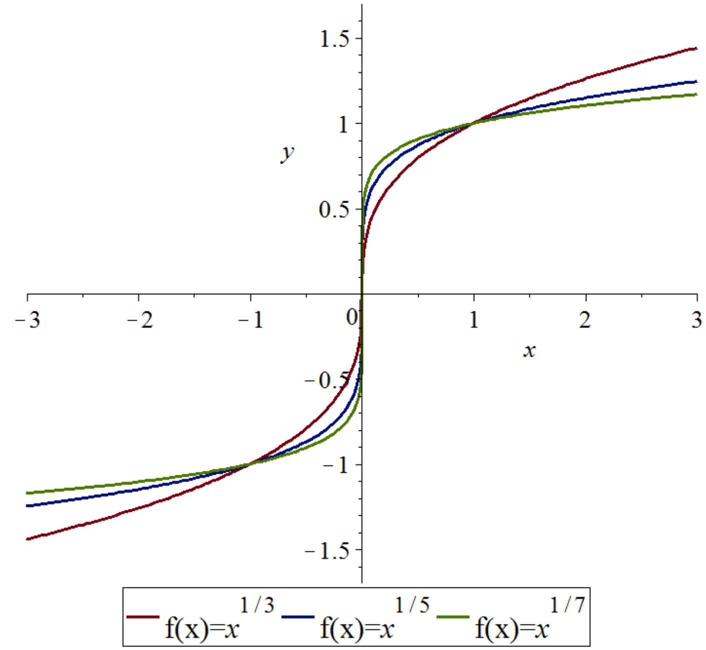


Domain: $x \neq 0$
Image: $y > 0$

Power functions (III) (ベキ関数)　$x \mapsto x^{a}$



<u>Case 3:</u> $a = \frac{1}{2}, \frac{1}{4} \ldots$
(surd or n-th root function, n even – 偶数)
The domain of definition（定義域） of f is
$\mathbb{R}_+ = [0, +\infty)$



<u>Case 4:</u> $a = \frac{1}{3}, \frac{1}{5}, \ldots\ldots$
(surd or n-th root function, n odd – 奇数)
The domain of definition（定義域） of f is $[0, \infty)$ or $\mathbb{R}$ (surd)

Power functions (IV) (ベキ関数) $x \mapsto x^{a}$

Case 5: $a = \frac{2}{3}, \frac{3}{4} \dots$ ($a \in \mathbb{Q}$)

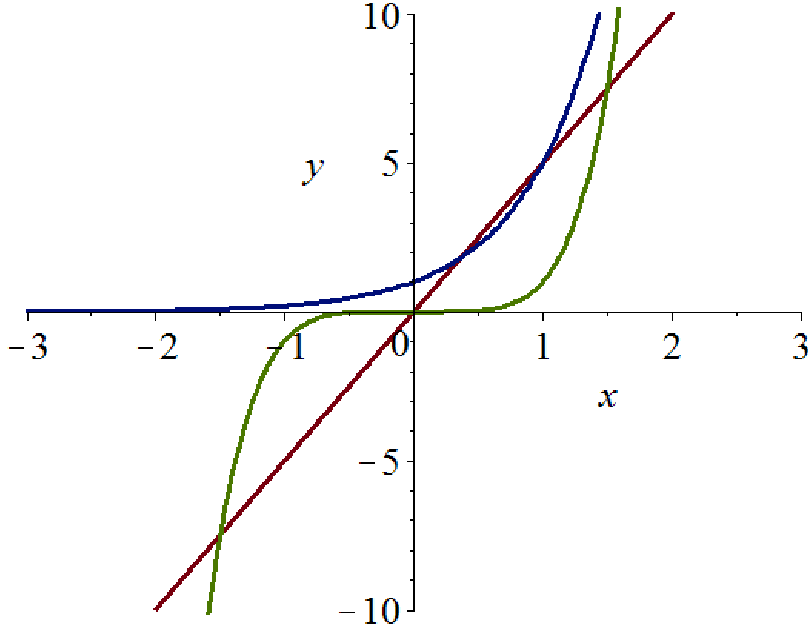
Ex: $x^{\frac{3}{2}} = (\sqrt{x})^3$ $\quad$ $x^{\frac{2}{3}} = \left(x^{1/3}\right)^2$

Case 6: $a \in \mathbb{R}$
Needs exponential and logarithm functions to be defined
Cf Slide 18)
（指数関数と対数関数に基づいて定義される。）

Exercise

- Match each equation with its graph:

  a) $y = 5x$ b) $y = 5^x$ c) $y = x^5$

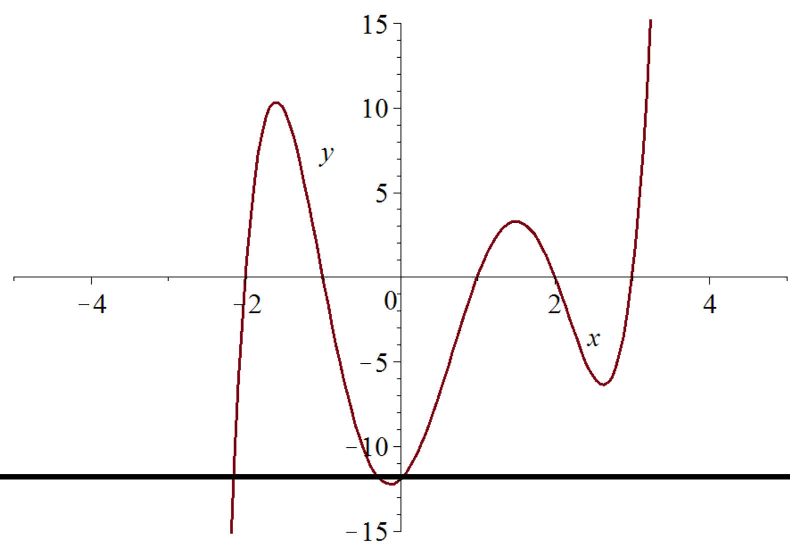


Polynomial functions（多項式関数）

- $f: x \mapsto a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$
  $(a_0, a_1 \cdots, a_n)$ are constants in $\mathbb{R}$ called coefficients（係数）

- a_n: leading coefficient （最高次係数 or 頭項係数）
- a_0: constant term（定数項）
- n is the degree（次数） denoted $\deg(f) = n$

Example: $f(x) = x^5 - 3x^4 - 5x^3 + 15x^2 + 4x - 12$



Remark: This polynomial has 5 real roots （実根). Since its degree is 5, it cannot have 6 roots.

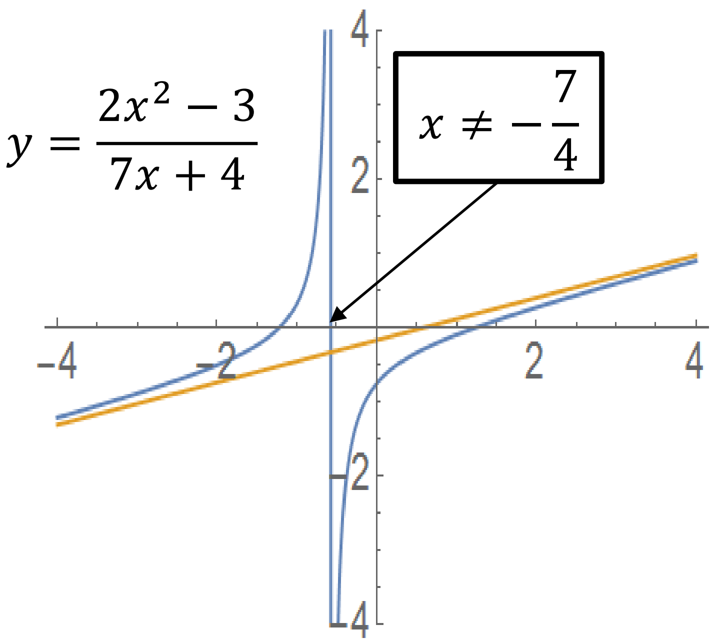
Exercise

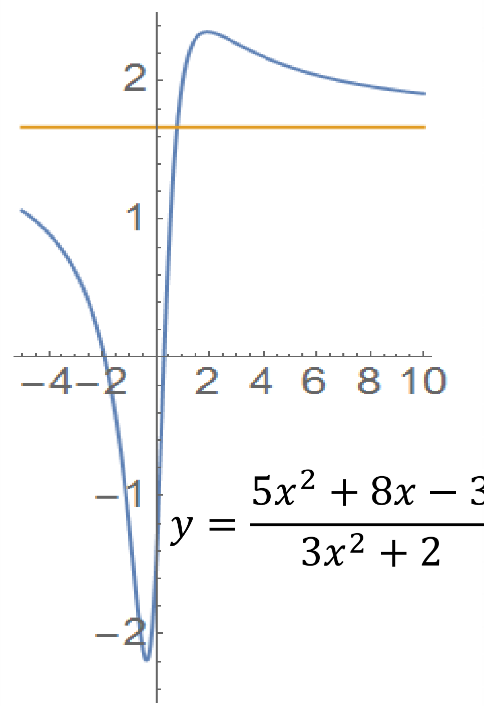
- Given 3 points $A(x_A, y_A), B(x_B, y_B), C(x_C, y_C)$
  not in a line,
  and such that $y_A \neq y_B,\ y_A \neq y_C$ and $y_B \neq y_C$
  there is one and only one parabola (放物線) that goes through the 3 points A, B, C.

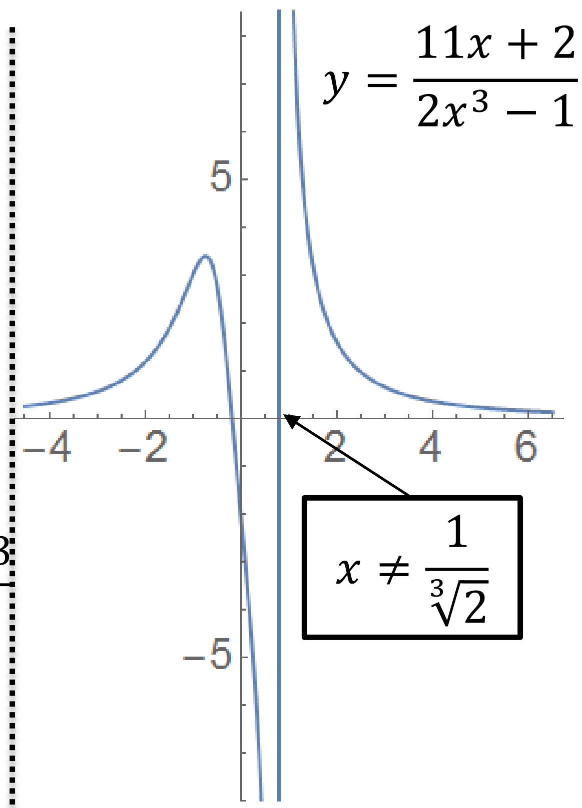
- **Exercise**: find the equation of the parabola going through 3 points (interpolation 補間):
  $A(0,0), \quad B(1,0), \quad C(2,1\ )$

Rational Functions (有理関数)

- $f: x \mapsto \dfrac{a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0}{b_m x^m + \cdots + b_1 x + b_0} = \dfrac{p(x)}{q(x)}$
- Domain of definition (定義域): $\{x \in \mathbb{R} : q(x) \neq 0\}$.







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Exponential function (指数関数)

- $f: x \mapsto e^x$ (or $\exp(x)$)
  Domain: $\mathbb{R}$ Image: $\mathbb{R}_{>0}$

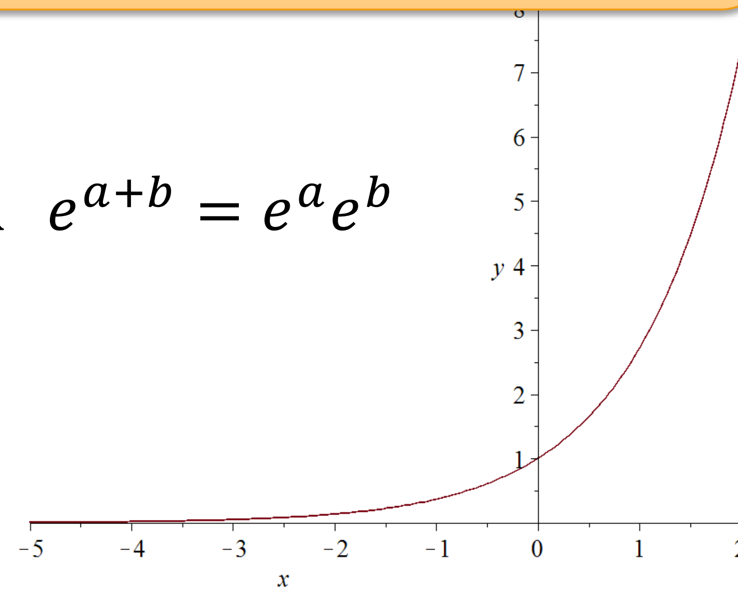
Notation（記号）

- This is the sole（唯一） function verifying:

$$\begin{cases} f(1) = e \approx 2.76 \\ f(a+b) = f(a)f(b) \end{cases}$$

(+ graph of f is smooth, 連続的)

特性化



- Properties
  - $e^0 = 1$, for all $a, b \in \mathbb{R}$ $e^{a+b} = e^a e^b$
  - $e \approx 2.76 \ldots$
  - $e^x < 1 \Leftrightarrow x < 0$

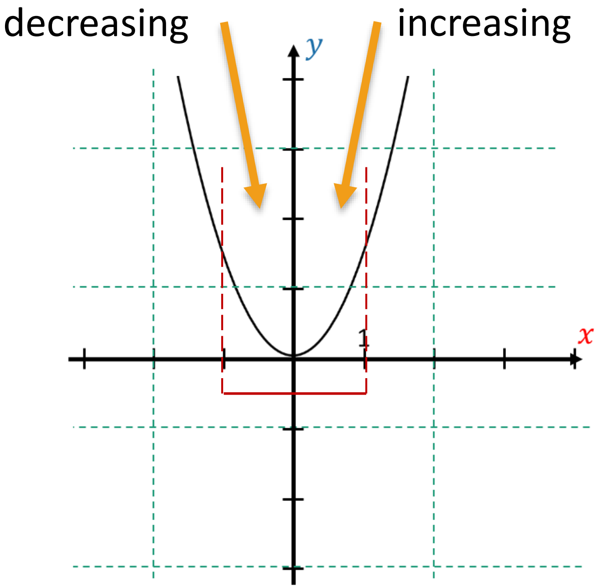
Inverse functions (逆関数)

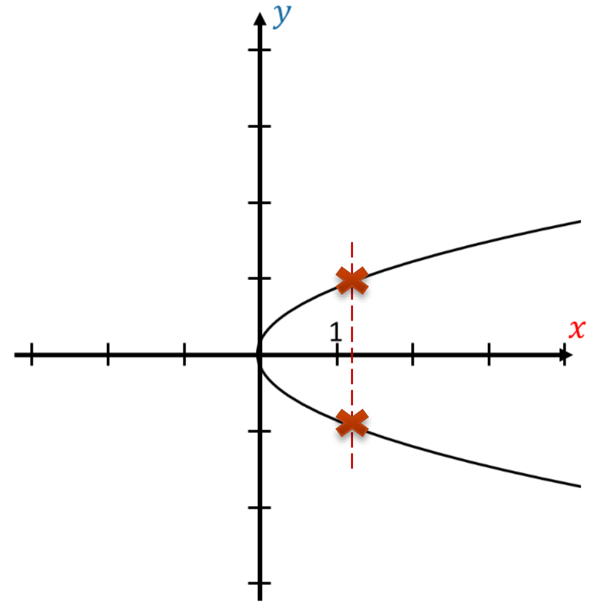
- $f: E \subset \mathbb{R} \to \mathbb{R}$
  Definition (定義) : The inverse function（逆関数）on the interval (区間) F is a function $f^{-1}: F \subset f(E) \to E$ such that: $f\left(f^{-1}(x)\right) = x$ for all $x \in F$.

- 注: Inverse functions do not necessarily exist（次のページを参照する）

  Ex: $f(x) = x^2,\ F = [0, +\infty),\quad f^{-1}(x) = \sqrt{x}$

- (**Existence theorem** 存在定理)
  If $f: E \subset \mathbb{R} \to \mathbb{R}$ is strictly increasing or decreasing (狭義単調減少 or 増加) on an interval $I \subseteq E$ then f admits an inverse function $f^{-1}: f(I) \to I$





$y = x^2$ has two solutions, $x = \pm\sqrt{y}$
$\Rightarrow$ This is the not the graph of a function.

The parabola $y = x^2$ is decreasing and then increasing on the interval $[-1,1]$

Therefore, $f(x) = x^2$ is not invertible (可逆でない) on the interval $[-1,1]$

Exercise

- Show that if f is strictly increasing on an interval $E \subset \mathbb{R}$ then its inverse function f^{-1} is also strictly increasing on $f(E)$.
  もしもfは区間Eで狭義単調増加であれば、その逆関数も区間$f(E)$で狭義単調増加であることを示せ。

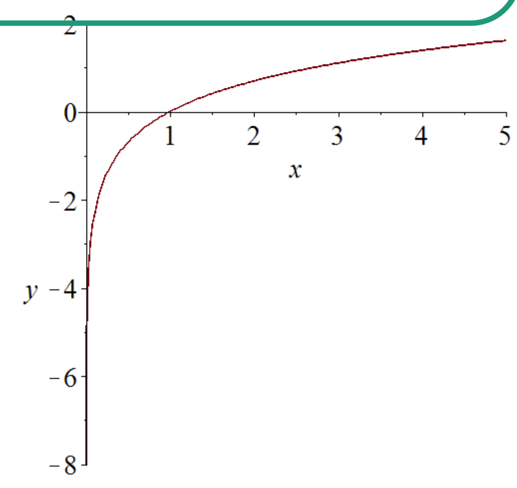
Logarithm（対数）

- $f: x \mapsto e^x$ is strictly increasing on $\mathbb{R}$, therefore it has an inverse function on its image $(0, \infty) = f(\mathbb{R})$.

- **Definition**: $f^{-1}: (0, \infty) \to \mathbb{R}$ is denoted ln and called "natural logarithm" (自然対数).

$$\ln(e^x) = x = e^{\ln(x)}$$

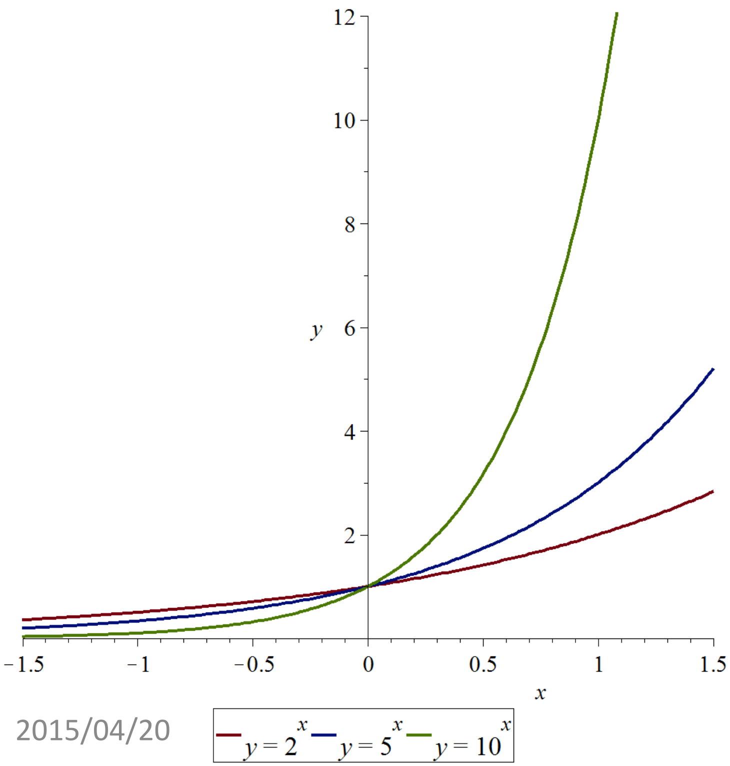
- $\ln(ab) = \ln(a) + \ln(b)$
  $\ln(1) = 0$

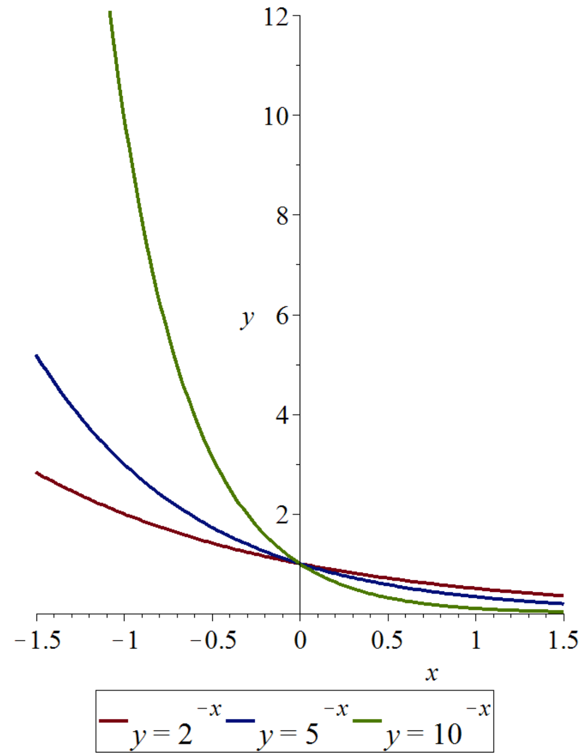


Other exponential functions

$a > 0$, $f: x \mapsto a^x$ is defined over $\mathbb{R}$ by: $a^x = e^{x \cdot ln(a)}$

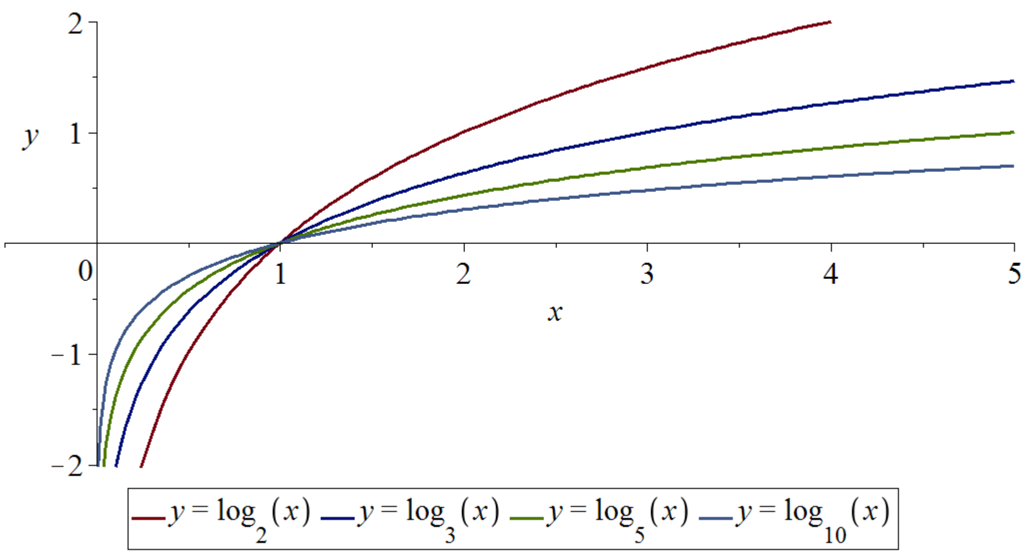
- The image $f(\mathbb{R})$ of f is $(0, +\infty)$.
- $a^{x+y} = e^{(x+y)\ln(a)} = e^{xln(a)}e^{yln(a)} = a^x a^y$





Other logarithmic functions

- f is strictly increasing.
  Therefore there is an inverse function $f^{-1}: (0, +\infty) \to \mathbb{R}$ denoted:
  $f^{-1}(x) = log_a(x)$ and called logarithm in basis a (低aをとする対数)
  $$\log_a(y) = x \iff y = a^x$$
- The logarithm in basis 10 is called "common logarithm" (常用対数)
- $\log_a(x + y) = \log_a(x) \log_a(y), \log_a(x^y) = y\log_a(x)$
- $\log_a(x) = \frac{\ln(x)}{\ln(a)}$



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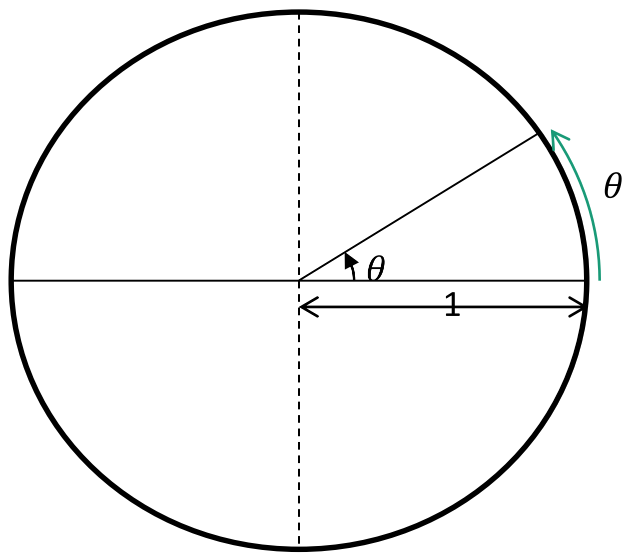
Trigonometric functions (三角関数)

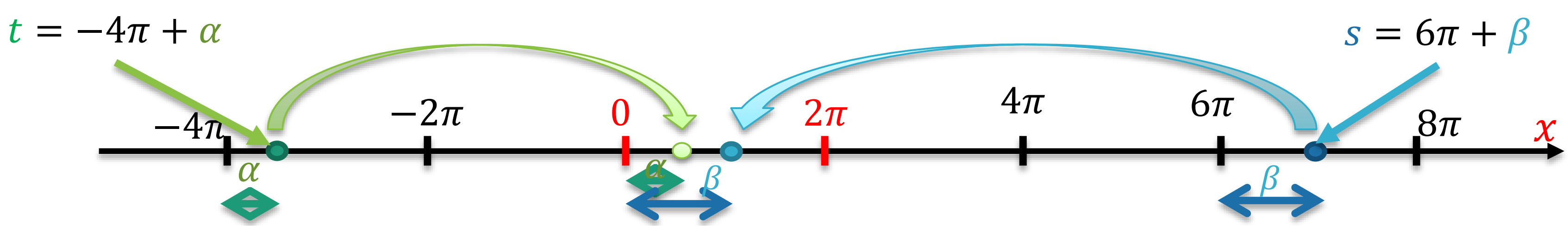


- Circumference (円周) of the unit circle (単位円)

$$2\pi r = 2\pi$$

- Radian (ラジアン)
  Measuring angle (角の測定)with length of arc (円弧の長さ)

- Angle $\hat{\theta}$ ~ length of the arc θ:
  $0 \le \hat{\theta} < 2\pi$



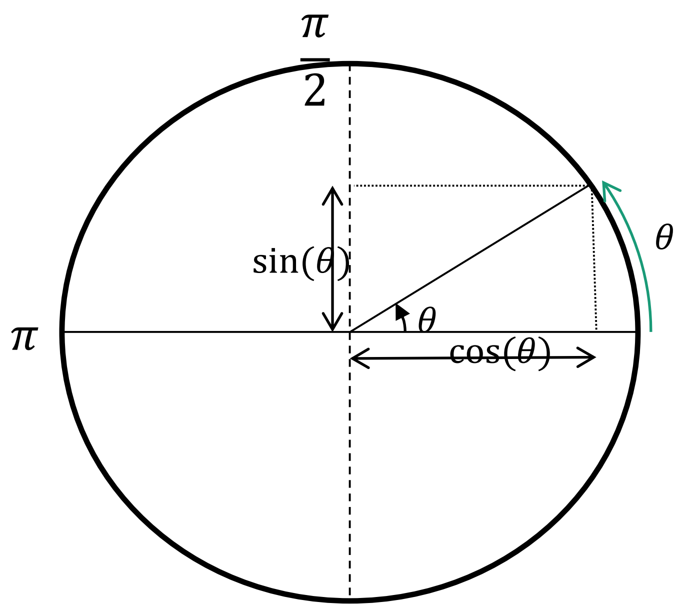
- α is the representative of t in $[0,2\pi)$ (written $t \equiv \alpha[2\pi]$)
- β is the representative of s in $[0,2\pi)$ (written $s \equiv \beta[2\pi]$)
- (αまたはβ は、$[0,2\pi)$の中で t またはsを代表する。)

Exercise

- Find the representative in $[0,2\pi)$ of the following angles ($\pi \approx 3.14$):

$$-\pi\ , \qquad -\frac{5\pi}{2}, \qquad 10$$

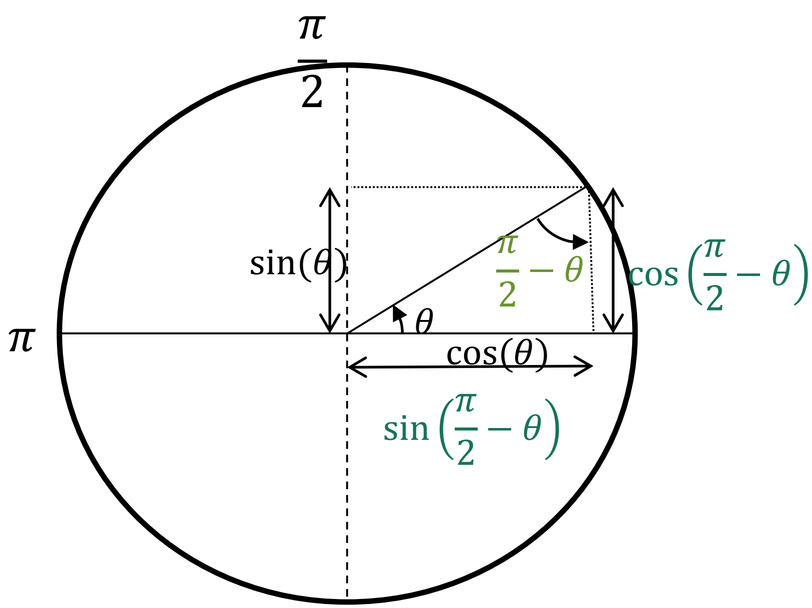
Sine and Cosine

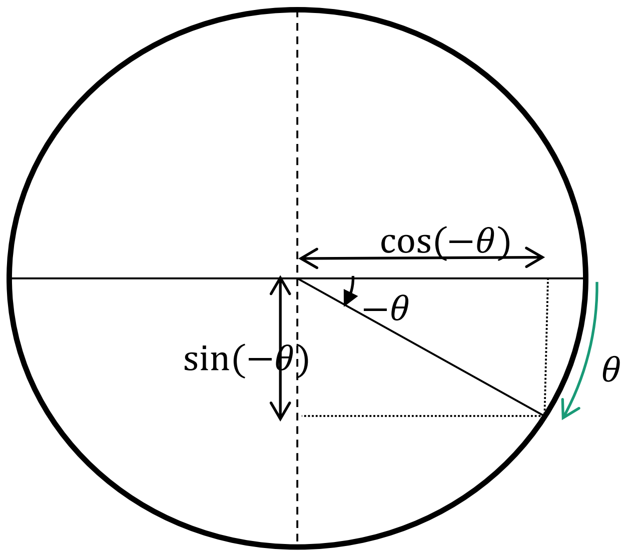


- $\cos(2\pi + \theta) = \cos(\theta)$
  $\sin(2\pi + \theta) = \sin(\theta)$
  (2π-periodic function 2π-周期関数)
- Pythagoras's theorem (ピタゴラスの定理)

$$\cos(\theta)^2 + \sin(\theta)^2 = 1$$

Sine and cosine (II)

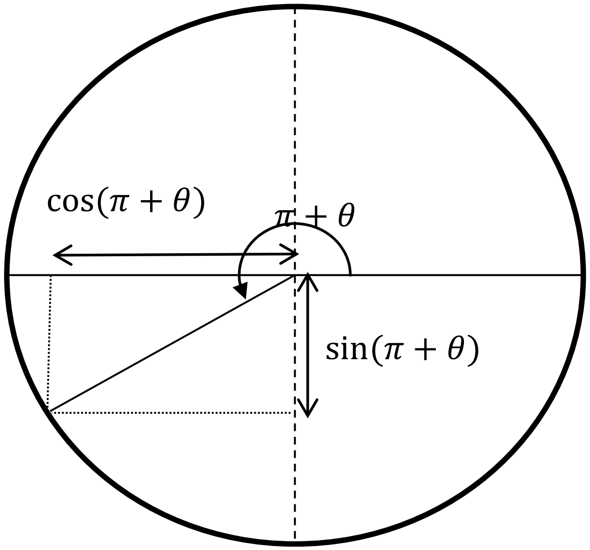


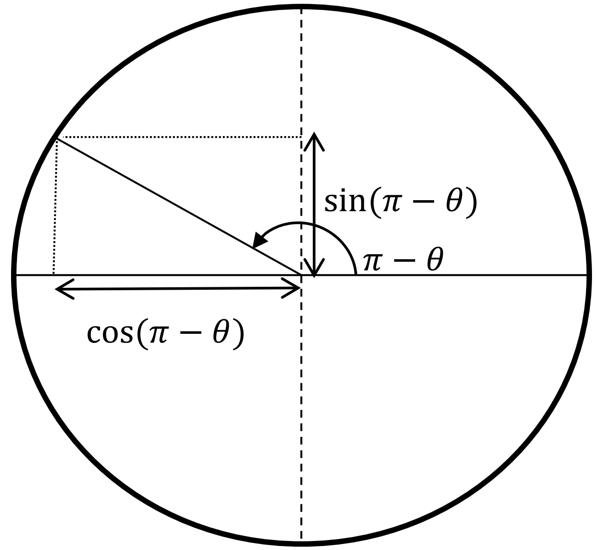


- $\cos\left(\frac{\pi}{2} - \theta\right) = \sin(\theta)$
- $\sin\left(\frac{\pi}{2} - \theta\right) = \cos(\theta)$

- $\cos(-\theta) = \cos(\theta)$
  $\sin(-\theta) = -\sin(\theta)$

Sine and Cosine (III)





- $\cos(\pi + \theta) = -\cos(\theta)$
- $\sin(\pi + \theta) = -\sin(\theta)$

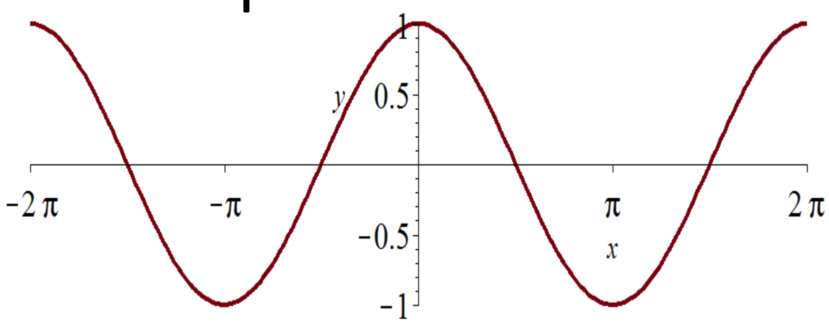
- $\cos(\pi - \theta) = -\cos(\theta)$
- $\sin(\pi - \theta) = \sin(\theta)$

Sine and Cosine (IV)

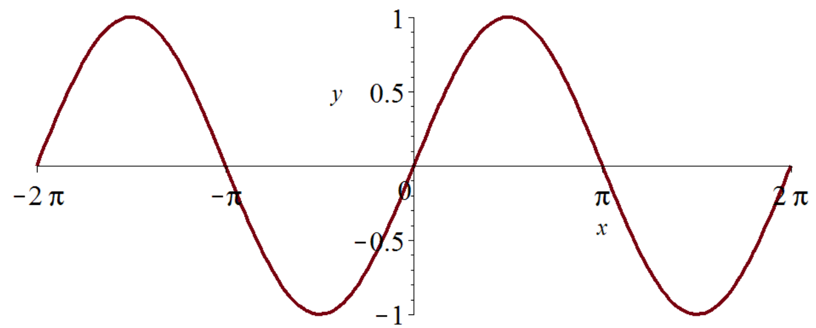
- Special values:

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\cos(\theta)$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\sin(\theta)$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1

- Graphs:



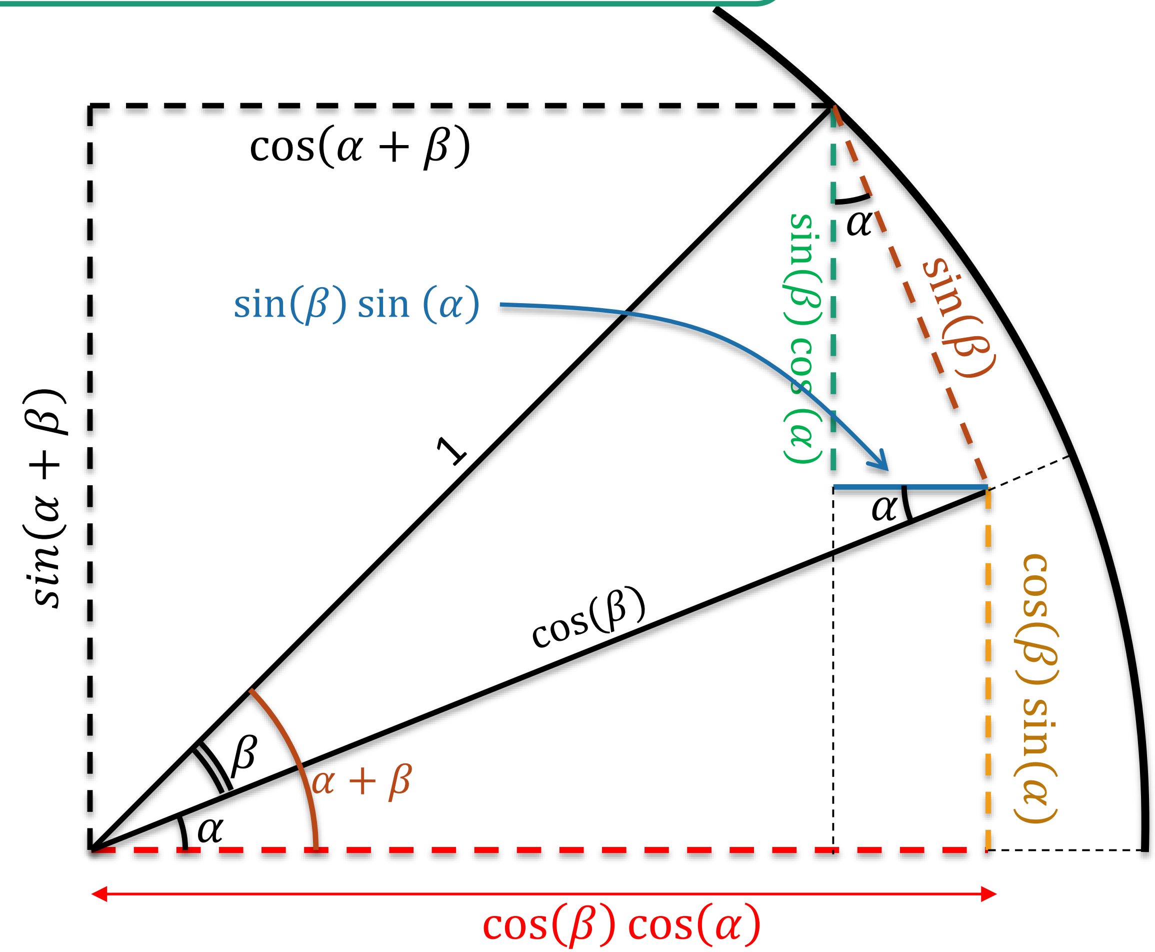
$y = \cos(x)$



$y = \sin(x)$

Angle Addition Formula

- $\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$
- $\sin(\alpha+\beta) = \cos(\alpha)\sin(\beta) + \sin(\alpha)\cos(\beta)$



Exercise

- Prove the following formulas
  - Double angle formulas（倍角公式）
    $$\sin(2\theta) = 2\sin(\theta)\cos(\theta)$$
    $$\cos(2\theta) = \cos(\theta)^2 - \sin(\theta)^2$$
  - Half angle formulas (半角の公式）
    $$\sin(\theta)^2 = \frac{1 - \cos(2\theta)}{2}$$
    $$\cos(\theta)^2 = \frac{1 + \cos(2\theta)}{2}$$

Exercise (II)

- What is the value of $\sin\left(\frac{7\pi}{12}\right)$?
  (Hint: $\frac{7\pi}{12} = \frac{\pi}{3} + \frac{\pi}{4}$)
- Evaluate $\cos\left(\frac{\pi}{12}\right)$?

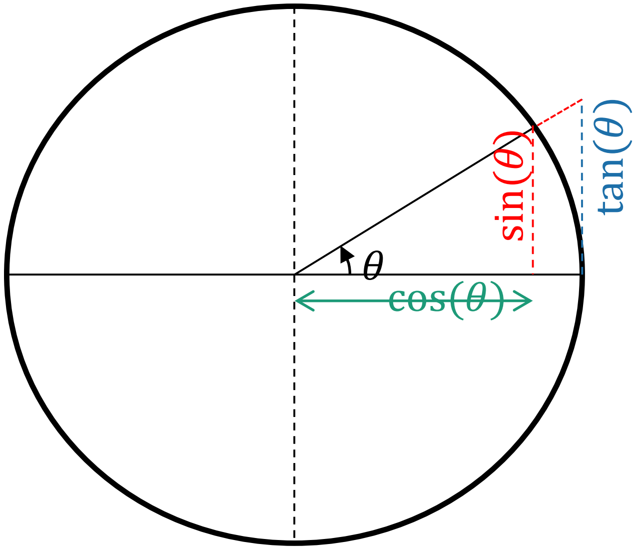
- Evaluate $\cos\left(\frac{\pi}{8}\right)^2$
  (Hint: Use of half-angle formulas)

Exercise (III)

- Solve the trigonometric equation with $0 \leq \theta < 2\pi$.

$$\sin(\theta)^2 = \frac{3}{4} \ , \qquad \sin(2\theta) - \cos(\theta) = 0$$

Tangent function



- Intercept theorem:
  $$\frac{\sin(\theta)}{\tan(\theta)} = \frac{\cos(\theta)}{1}, \qquad \theta \neq \frac{\pi}{2}\ [\pi]$$
- $$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}, \qquad \theta \neq \frac{\pi}{2}\ [\pi]$$
- tan is π-periodic (π-周期)

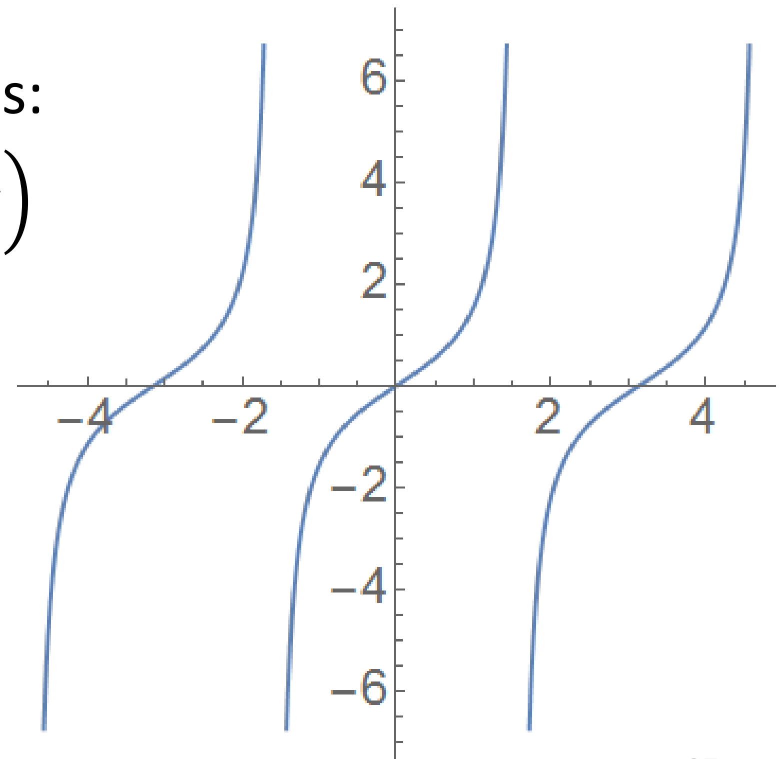
- $\tan(\alpha+\beta) = \frac{\sin(\alpha)\cos(\beta)+\sin(\beta)\cos(\alpha)}{\cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta)}$
- $= \frac{\sin(\alpha)\cos(\beta)+\sin(\beta)\cos(\alpha)}{\cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta)} \left(\frac{\cos(\alpha)\cos(\beta)}{\cos(\alpha)\cos(\beta)}\right)$
- $\tan(\alpha+\beta) = \frac{\tan(\alpha)+\tan(\beta)}{1-\tan(\alpha)\tan(\beta)}$

Tangent function

θ	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$
$\cos(\theta)$	1	$\sqrt{3}/2$	$\sqrt{2}/2$	$1/2$	0
$\sin(\theta)$	0	$1/2$	$\sqrt{2}/2$	$\sqrt{3}/2$	1
$\tan(\theta)$	0	$\sqrt{3}/3$	1	$\sqrt{3}$	$-$

- The domain of definition of tan is:

$$\bigcup_{k\in\mathbb{Z}}\left(-\frac{\pi}{2}+k\pi\,,\frac{\pi}{2}+k\pi\right)$$



Composition of functions（関数の合成）

- Let $f: E \mapsto \mathbb{R}$, $g: F \mapsto \mathbb{R}$ be two functions, such that the image $g(F)$ is contained in the domain of definition E of f : $g(F) \subset E$.

- The function
$$f \circ g(x) = f\big(g(x)\big)$$
is called the composite of f with g
(Reading: "f composed with g")

Ex: $f(x) = x^2$, $g(x) = \sqrt{x}$, then $f \circ g(x) = (\sqrt{x})^2 = x$.

注: the domain of $f \circ g$ is not $\mathbb{R}$ but $[0, +\infty)$!

Homework: Hand in on April 27th ,pliz

- Find a formula for $f \circ g \circ h$:
  - $f(x) = \sqrt{x+1}$, $g(x) = \frac{1}{x+4}$, $h(x) = \frac{1}{x}$.

- Fill in the following table (次の表を完成せよ)

$g(x)$	$f(x)$	$f \circ g(x)$
$x - 7$	$\sqrt{x}$	?
?	$\sqrt{x-5}$	$\sqrt{x^2 - 5}$
?	$1 + \frac{1}{x}$	$\frac{1}{\cos^2(x)}$
?	$\sqrt{x}$	$\lvert x \rvert$

- Review Mathematica functions and watch the video:
  http://www.wolfram.com/broadcast/video.php?c=86&v=306 (日本語)
  http://www.wolfram.com/broadcast/video.php?c=86&v=307 (English)