Essential Mathematics for Global Leaders I

「みがかずば」の精神に基づきイノベーションを創出し続ける

理工系グローバルリーダーの育成

Xavier DAHAN
Ochanomizu Leading Promotion Center
Office:理学部2号館503
mail: dahan.xavier@ocha.ac.jp

文部科学省採択プロジェクト
お茶の水女子大学
博士課程教育リーディングプログラム

イノベーション創成科目

物理	数学	情報

理工学の基盤となる知識

- e-Physics
- e-Chemistry
- e-Mathematics
- e-Bioinformatic
- e-Computer Science
- e-Engineering and Technology

イノベーション創成に向けた幅広い知見

e- はessentialの略　　すべて英語で実施する

Applied mathematics Ｉ（応用数学）

- (mathematic) Modelization:
  1. select meaningful numerical parameters to describe phenomena, experiments etc...
     ある現象あるいは実験などを記述するために有意義な数値パラメータを選択し、
  2. Use mathematical equations to study how these parameters are related.
     このパラメータの相互関係を検討するために数学方程式を用いる。

- Optimization（最適化）
  - Many concrete problems can be expressed as: minimization of a function (cost, length etc.) under constraints
    さまざまな具体的な問題が次の通りに表される：制約付き最小化
  - 例：$\min \quad costF$(労働コスト、生産コスト、原料コスト、...)
    example of constraint: client requirements: time, nbr of orders, etc...
    （制約の例：クライアント要求→時間、注文数、など。）

Applied mathematics II（応用数学）

- Statistics （統計）
  - Find a simple pattern/meaning/interpretation in many data.
    多くのデータの中、パターン・意味・解釈を見つける。

- Mathematics for Computer Science（コンピューターサイエンスにおける数学）
  - Because computers can only process finite numbers & data, discrete mathematics occur a lot: data transmission (error-correcting code), cryptogaphy, …
    計算機は有限な数のデータだけを処理できるため、離散数学がよくある: データ転送（誤り訂正符号）、暗号理論など
  - Signal processing（信号処理）: audio/video-communication, data compression, speech recognition
    (データ圧縮、音声認識、など）
  - Algorithm design and analysis（アルゴリズム設計、解析など）

Basic concepts of modelization

1. Predictive model: from the model to the real world
（予測的：モデルから現実の世界へ）
    - Example: meteorology. But all physical equations (Newton, Maxwell, Einstein etc.)
天気予報。これだけでなく、すべての物理学的な方程式（ニュートン、マクスウェル、アインシュタインなど）。

2. Descriptive: from real world to the model
（記述的：現実の世界からモデルへ）
    - Example: financial data (accounting)
財務データ（会計処理）。

Evolutionary process and predictive model（進化の過程と予測的なモデル）

- Time is very often a main parameter.
  時間は主要なパラメターの一つであることが非常に多い。
  - Predictive model are often used to describe evolutionary process
- Most evolutionary processes are modeled by differential equations.
  ほとんどの進化の過程が微分方程式によってモデル化されている。

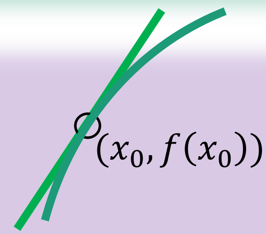
→　One aim of this course is to introduce and understand basic of differential equations.
　本科目の目的としては、基礎（常）微分方程式を紹介し、理解することである。

Path towards Diff. Eq.（微分方程式への道）

1. Review of basic mathematics
   基礎の数学を復習する

2. Differential calculus（微分）→

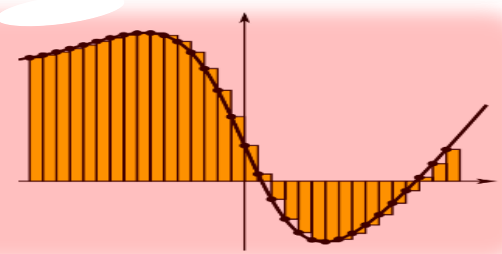
   $$f'(x_0) = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$$



$f'(x_0)$→ slope of the tangent （接線の傾き）

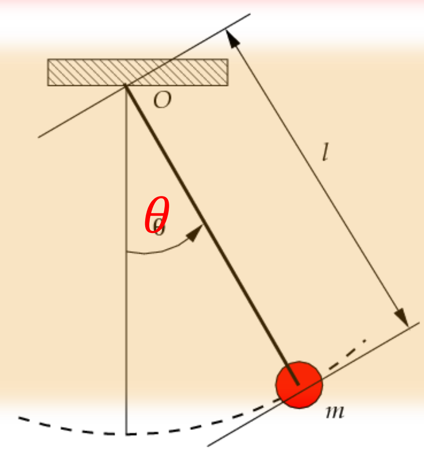
3. Integral calculus （積分）→

   $$\int_a^b f(x)\,dx = \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} f\left(a + k\frac{b-a}{n}\right)$$



4. Ordinary differential equations （常微分方程式）→

   $$\frac{d^2\theta}{dt^2} + \frac{g}{\ell}\sin\theta = 0$$



Plan (tentative)

[4/13] L1 : introduction. Review of high-school mathematics in English.

[4/20-27] L2-3 : Functions and graphs. Plotting with Mathematica I (グラフをプロットする)

[5/7] L4 : Infinitely small and large : limits (極限)

[5/11] L5 : Continuity and differentiation (連続性と微分法)

[5/18] L6 : Differentiation II : extrema, related rates ...（極値と...）

[5/25] L7 : Differentiation III : Newton's method, Taylor's expansion (ニュートン法とテイラー展開）

[6/1] L8 : Mid-term test. Integration I : definition, fundamental theorem of calculus　積分I.

[6/8] L9 : computation of indefinite integrals　不定積分

[6/15] L10 : Application of Integration I : length, volume and surfaces
積分の応用：長さ、面積、体積

[6/22] L11 : Application of Integration II : average, center of mass （質量中心）, work of a force.

[6/29] L12 : Ordinary Differential equations (one variable)　常微分方程式

[7/6] L13 : Linear Differential Equations of order 2 : harmonic oscillators (small-angle pendulum, spring). 二階線形微分方程式：調和振動子　(振幅が小さい振り子、ばね)

[7/13] L14 : Ordinary Differential Equations with Mathematica. Mathematicaを利用して常微分

Some English maths terms…

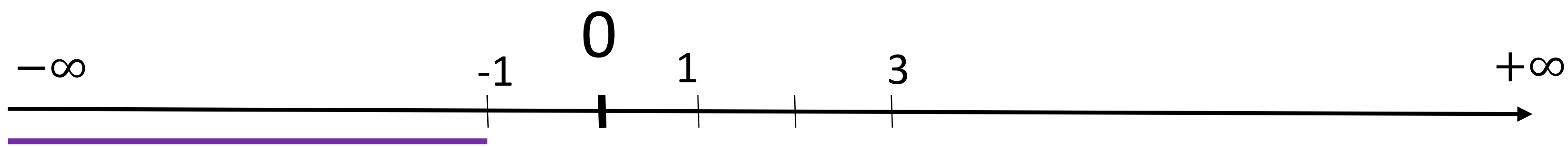
- { } denotes a set of objects:
  はモノの集合を現す。

  Ex: $\{\dots,-2,-1,0,1,2,\dots\}=\mathbb{Z}\rightarrow$ set of integer numbers（整数の全体）
  Even number (偶数) → $\dots,-6,-4,-2,0,2,4,6,\dots$
  Odd number (奇数）→ $\dots,-5,-3,-1,1,3,5,\dots$

- $-3\in\mathbb{Z}$,
  Reading: "minus three belongs to the integers"
  マイナス三は整数に含まれる。 Or
  "minus three is in Z" (マイナス三は$\mathbb{Z}$にある）

- $\mathbb{R}$ is the set of real numbers. $-1,2,\sqrt{2},\pi\in\mathbb{R}$
  は実数全体である。

- The four fundamental arithmetic operations: 四則演算
- Addition, subtraction, multiplication, division. 足し算・加法、 引き算・減法、 かけ算・乗法、 割り算・除法



- $[1,3]$ "closed interval" $= \{a \in \mathbb{R} : 1 \leq a \leq 3\}$
  閉区間
  $(1,3)$ "open interval" $= \{a \in \mathbb{R} : 1 < a < 3\}$
  開区間

"Such that"
(ただし)
条件を示す記号

- $S = \{a \in \mathbb{R} : a \leq -1\} = (-\infty, -1]$

Reading: set of real numbers which are smaller or equal than minus one.

- $\mathbb{R} = (-\infty, +\infty)$

Today's agenda 今日の目次

1. Fractions 分数
2. Modulus (absolute values 絶対値)
3. Inequalities 不等式
4. Expansion and factorization 展開と分解
   Binomial expansion 二項定理
5. Square root and n-th root 乗根
6. Quadratic Equations ２次方程式
7. Summation of arithmetic & geometric progressions 等差と等比数列の和

Fractions (分数)

- $\frac{a}{b}$ where $a, b \in \mathbb{R}$, $b \neq 0$.

(Reading: "a over b". 1/3→one third. ¼→ "a quarter", or "one fourth" or "one over four"...)

- a is called the numerator (分子）
  b is called the denominator (分母)

- Very often, $a, b \in \mathbb{Z}$. Then, we define the:
  Rational numbers (有理数)

$$\mathbb{Q} = \{\frac{a}{b} : \ a, b \in \mathbb{Z}, \quad b \neq 0\}$$

Ex: $-\frac{1}{3} \in \mathbb{Q}$, but $\sqrt{2}, \sqrt{3}, \pi \notin \mathbb{Q}$ (Irrational numbers: 無理数）

Operations on fractions (分数の演算)

1. $\frac{a}{c}+\frac{b}{c}=\frac{a+b}{c}$, $(\mathrm{c}\neq 0)$ (Same denominator addition)
(共通分母の和)

2. $\frac{a}{b}+\frac{c}{d}=\frac{ad+bc}{bd}$ $(b,d\neq 0)$ Ex: $\frac{1}{3}+\frac{1}{6}=\frac{2}{6}+\frac{1}{6}=\frac{3}{6}=\frac{1}{2}$
(Reduction to the same denominator. 通分する)
(Reading: "a over b, plus c over d")

3. $\frac{a}{c}\times\frac{b}{d}=\frac{ab}{cd}$, $(c,d\neq 0)$ Ex: $\frac{3}{4}\times\frac{2}{3}=\frac{1}{2}$
(Reading: "a over c times b over d")

4. $\frac{a}{b}\ /\ \frac{c}{d}=\frac{a}{b}\times\frac{d}{c}=\frac{ad}{bc}$ $(b,c,d\neq 0)$ Ex: $\frac{3/4}{2/3}=\frac{3}{4}\times\frac{3}{2}=\frac{9}{8}$
(division of fractions is multiplication by the inverse)
(分数での割り算はその法数の逆数による積である)

Modulus (absolute value 絶対値)

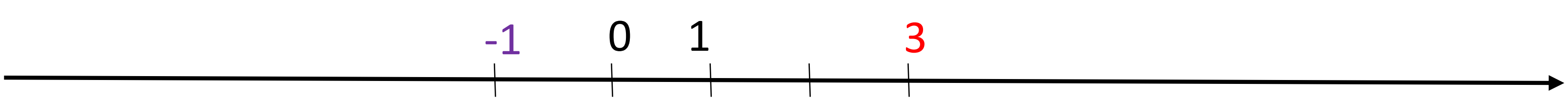
- $a \in \mathbb{R}$, the absolute value of a (or modulus) is:
  - $|a| = \begin{cases} a & \text{if } a \geq 0 \text{ (positive, 正数)} \\ -a & \text{if } a < 0 \text{ (negative, 負数)} \end{cases}$

- Ex: $\left|-\sqrt{2}\right| = -\left(-\sqrt{2}\right) = \sqrt{2}$ (Reading: Absolute value of minus square root of 2 is equal to square root of 2)

· (dot) means multiplication

- For any $x, y \in \mathbb{R}$, $|x \cdot y| = |x| \cdot |y|$ ($\text{Ex: } |-3 \cdot 4| = |-3||4| = 12$)
  - $|x| = |-x|$ (Ex: |-3|=|3|=3)
  - $|x + y| \leq |x| + |y|$ (Triangle inequality, 三角不等式)

- Distance (距離) between 2 points $a, b \in \mathbb{R}$: $|a - b|$
  Ex: distance from point 3 to point -1 is $|3 - (-1)| = 4$



Inequality equation (不等式)

- Given a formula $f(x)$ depending on an unknown x. Find the values of x that satisfy $f(x) \geq 0$ (or > 0)
  - Ex: $\left|\frac{3x-1}{4}\right| \geq 2$ (Reading: "Absolute value of 3x-1 over 4 greater or equal than 2")

- Rules for solving inequalities: ($y, z \rightarrow$ unknowns, $a \in \mathbb{R}$)
  1. $|y| \leq a \Leftrightarrow -a \leq y \leq a$
  2. $|y| > a \Leftrightarrow y > a$ or $y < -a$
     (Ex: $\frac{3x-1}{4} \geq 2$ or $\frac{3x-1}{4} \leq -2$)
  3. $y > z \Rightarrow$ ($ay > az$ if $a > 0$) and ($ay < az$ if $a < 0$)
     (Ex: $3x - 1 \geq 8$ or $3x - 1 \leq -8$)
  4. $y > z \Rightarrow (a + y > a + z)$
     (Ex: $3x \geq 9$ or $3x \leq -7 \Rightarrow \boxed{x \geq 3}$ or $\boxed{x \leq -\frac{7}{3}}$)

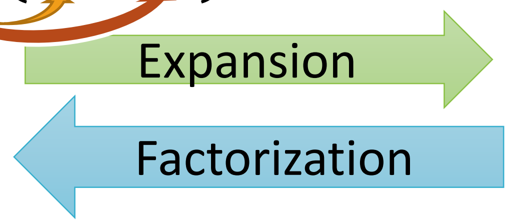
Exercise

- Find the solution set of the inequality equation:

$$|x - 4| > 1$$

Expansion and factorization (展開と分解)

1. $(a+b)(c+d) = ac + ad + bc + bd$



2. $(a \pm b)^2 = (a^2 \pm 2ab + b^2)$ (How much is 101^2 ?)
3. $a^2 - b^2 = (a-b)(a+b)$
4. $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$

- <u>Exercise: </u>Factorize $a^n - b^n$

Binomial expansion（二項定理）

- Binomial ? polynomial with two terms: $x + a$.
- How to expand $(x+a)^2, (x+a)^3, \ldots, (x+a)^n$.

$$(x+a)^0 = 1$$
$$(x+a)^1 = x + a$$
$$(x+a)^2 = x^2 + 2ax + a^2$$
$$(x+a)^3 = x^3 + 3ax^2 + 3a^2x + a^3$$
$$(x+a)^4 = x^4 + 4ax^3 + 6a^2x^2 + 4a^3x + a^4$$
$$(x+a)^5 = x^5 + \_\_ax^4 + \_\_a^2x^3 + \_\_a^3x^2 + \_\_a^4x + a^5$$

Can you see a pattern?

Can you make a guess what the next one would be?

- A simple pattern for the x's and a's coefficient. What about the other coefficients ?

Binomial expansion (II)

$$(x+a)^5 = 1x^5 + 5ax^4 + 10a^2x^3 + 10a^3x^2 + 5a^4x + 1a^5$$

$$(x+a)^0 = 1$$

$$(x+a)^1 = 1x + 1a$$

$$(x+a)^2 = 1x^2 + 2ax + 1a^2$$

$$(x+a)^3 = 1x^3 + 3ax^2 + 3a^2x + 1a^3$$

$$(x+a)^4 = 1x^4 + 4ax^3 + 6a^2x^2 + 4a^3x + 1a^4$$

1

1 + 1

1 + 2 + 1

1 + 3 + 3 + 1

1 + 4 + 6 + 4 + 1

1 5 10 10 5 1

Can you guess the next row?

Binomial expansion (III)

- Pascal's triangle(パスカル三角形）

Ex: $10 = C_5^2 = \dfrac{5\times4\times3\times2}{2\times3\times2}$

												Line
					1							0
				1		1						1
			1		2		1					2
		1		3		3		1				3
	1		4		6		4		1			4
1		5		10		10		5		1		5
0		1		2		3		4		5		

Column

- The term in column k and line n in Pascal's triangle is given by: $\dfrac{n!}{k!(n-k)!} = C_n^k$ (where $n!$ is read "n factorial" n 階乗 , $n! = n \times n-1 \times \cdots \times 2$)

- $(x+a)^n = x^n + nax^{n-1} + \cdots + C_n^k x^k a^{n-k} + \cdots + na^{n-1}x + a^n$

Exercise:

- Expand $(x + 1)^6$

- Coefficient of $c^3 d^7$ in the expansion of $(c + d)^{10}$

(Real) n-th root (surds 累乗根)

Definition: Let $a \in \mathbb{R}$. A real n-th root of a (if it exists) is a number $r \in \mathbb{R}$ such that $r^n = a$.

- It doesn't always exist if $a < 0$ (Ex: -1 has no 2-nd real root).
  もし$a < 0$ならば、必ずしもn-乗根が存在するとはかぎらない。
- **Ex**: 2 and -2 are square roots of 4.
  $\sqrt{2}$ and $-\sqrt{2}$ are 2-nd roots of 2.
  2 and -2 are 4-th root of 16.

$\sqrt{4} =$

$\sqrt[4]{16} =$

- Exercise: Is there a 3-rd real root of -8 ?

Definition: If $a > 0$, there is always a n-th root r, and only one satisfying: $r > 0$. It is denoted $\sqrt[n]{a}$ or $a^{1/n}$.
もし$a > 0$ならば、唯一のn-乗根$r > 0$が存在する。

Operations with n-th roots

1. $\sqrt[n]{a} \times \sqrt[n]{b} = \sqrt[n]{ab}$ Ex: $\sqrt{18} = \sqrt{9 \times 2} = 3\sqrt{2}$
2. $\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}$ Ex: $\frac{\sqrt[3]{6}}{\sqrt[3]{3}} = \sqrt[3]{\frac{6}{3}} = \sqrt[3]{2}\ (\simeq 1.26)$
3. $b\sqrt[n]{a} \pm c\sqrt[n]{a} = (b \pm c)\sqrt[n]{a}$ Ex: $3\sqrt{10} - 2\sqrt{10} = \sqrt{10}$

- **Note**: In a fraction with square roots, it is better that in the denominator, no square root appear:

$$\frac{a}{b+\sqrt{c}} = \frac{a(b-\sqrt{c})}{(b+\sqrt{c})(b-\sqrt{c})} = \frac{a(b-\sqrt{c})}{b^2-c}$$

Square root in the denominator: NG

No square root in the denominator: better

Exercise

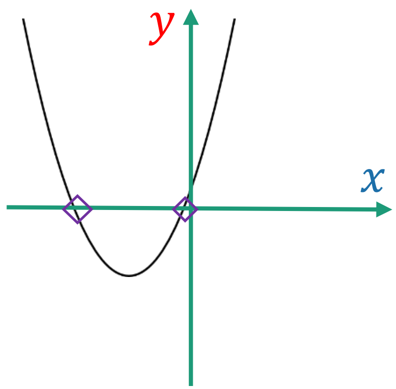
- Remove the square roots in the denominator:

$$\frac{5}{\sqrt{7}+\sqrt{5}}$$

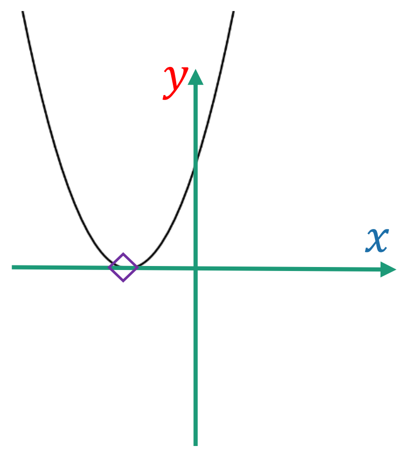
Quadratic equations (二次方程式）

- $ax^2 + bx + c = 0$ $\quad x \rightarrow$ unknown,
  $a, b, c \rightarrow$ (real) coefficients（実数の係数）

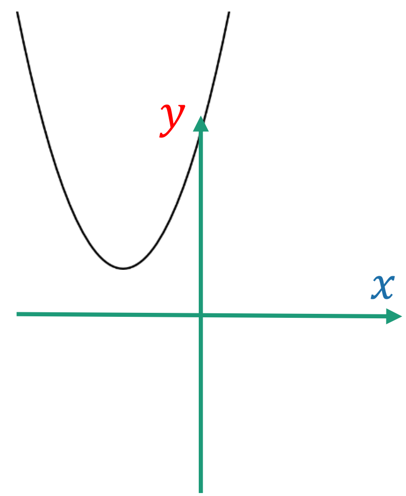
- The plot of the curve (曲線）$y = ax^2 + bx + c$ in the (x, y)-plane is called a parabola and there are three cases:



2 distinct roots
(相異なる根がふたつ)

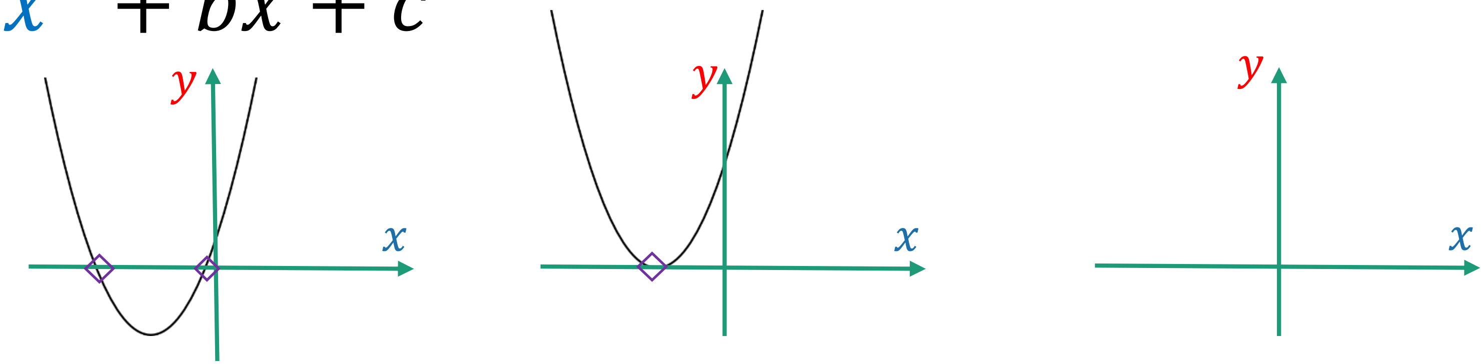


1 (multiple) root
(重根がひとつ)



No (real) root
実根なし

$$y = ax^2 + bx + c$$



2 distinct roots $\Delta > 0$

1 (multiple) root $\Delta = 0$

No (real) root $\Delta < 0$

- $\Delta = b^2 - 4ac$ (discriminant 判別式)
- If $\Delta \geq 0$, then $\sqrt{\Delta}$ exists and:

$$ax^2 + bx + c = a\left(x - \frac{b - \sqrt{\Delta}}{2a}\right)\left(x - \frac{b + \sqrt{\Delta}}{2a}\right)$$

therefore

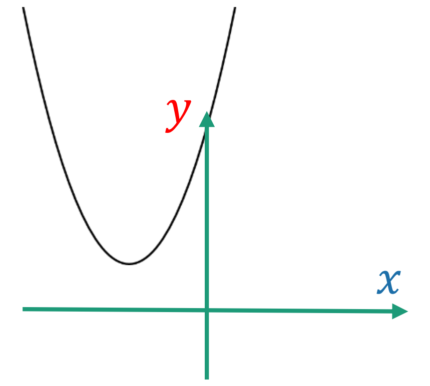
$$ax^2 + bx + c = 0 \Leftrightarrow a\left(x - \frac{b - \sqrt{\Delta}}{2a}\right)\left(x - \frac{b + \sqrt{\Delta}}{2a}\right) = 0$$

$\Leftrightarrow$ $\boxed{x = \frac{b - \sqrt{\Delta}}{2a}}$ or $\boxed{x = \frac{b + \sqrt{\Delta}}{2a}}$ (both are equal if $\Delta = 0$)

$y = ax^2 + bx + c.$

No (real) root

$\Delta < 0$



- If $\Delta < 0$, we cannot take $\sqrt{\Delta}$ so $ax^2 + bx + c$ has no solution in $\mathbb{R}$.
  Δの平方根をとれないため、 $\mathbb{R}$上の根がない

- Introduce the complex numbers.
  $\mathbb{C} = \{ \alpha + i\beta \mid \alpha, \beta \in \mathbb{R}, \ \ i^2 = -1\}$ (複素数全体)

- If $\Delta < 0$, then $-\Delta = i^2\Delta > 0$, and:

$$ax^2 + bx + c = 0 \Leftrightarrow a\left(x - \frac{b - i\sqrt{-\Delta}}{2a}\right)\left(x - \frac{b + i\sqrt{-\Delta}}{2a}\right) = 0$$

- **Example**: $x^2 + x + 1 = 0$ has a delta $\Delta = -3$
  Thus $\Delta = 3i^2$ and $\sqrt{\Delta} = i\sqrt{3}$.
  Its complex roots are $x_1 = -1/2 + i\sqrt{3}/2$
  and $x_2 = -1/2 - i\sqrt{3}/2.$

Exercise

1. Factorize the polynomial: $x^2 + x - 6$
2. Find the roots of the polynomial:
$$x^2 + 3x + 1$$
3. Find the roots of the polynomial
$$x^2 - x + 1$$

Summation (総和)

100

- $1 + 2 + 3 + \cdots + 97 + 98 + 99$

$$\sum_{k=1}^{99} k = 49 \times 100 + 50 = 99 \times \frac{100}{2}$$

100

$$\cdots + 48 + 49 + 50 + 51 + 52 \cdots$$

$$\sum_{k=1}^{n} k = 1 + 2 + \cdots + n = \frac{n \times (n+1)}{2}$$

Arithmetic progression's summation
等差数列の和

Arithmetic series (II)

- Ex: $35 + 40 + 45 + \cdots + 90 + 95 + 100$

$$\frac{\begin{array}{c}35 + 40 + 45 + \cdots + 90 + 95 + 100 \\ 100 + 95 + 90 + \cdots + 45 + 40 + 35\end{array}}{135 + 135 + 135 + \cdots + 135 + 135 + 135}$$

- Moreover

$$35 + 40 + \cdots + 95 + 100 = \sum_{k=0}^{13} 35 + 5k$$

- Therefore,

First term | Last term | Number of terms

$$2 \times \sum_{k=0}^{13} 35 + 5k = (35 + 100) \times 14 = 135 \times 14$$

- More generally:

First term: $k = 0$ | Last term: $k = n$

$$\sum_{k=0}^{n} a_0 + dk = \frac{\left(a_0 + (a_0 + nd)\right)(n+1)}{2}$$

Exercise

- Write the following sum

$$S = 17 + 20 + 23 + \cdots + 77 + 80$$

as follows (find n and d)

$$S = \sum_{k=0}^{k=n} 17 + dk$$

- Compute the sum S

Geometric series（等比数列）

- Remark: $(q+1)(q-1) = q^2 - 1$

$$(q^2+q+1)(q-1) = q^3 + q^2 + q - q^2 - q - 1$$
$$= q^3 - 1$$

$$(q^3+q^2+q+1)(q-1)$$
$$= q^4 + q^3 + q^2 + q + 1 - q^3 - q^2 - q - 1 = q^4 - 1$$

- More generally:

$$(q^n + q^{n-1} + \cdots + q + q)(q-1) = q^{n+1} - 1$$

- Therefore, if $q \neq 1$

$$1 + q + q^2 + \cdots + q^n = \frac{q^{n+1} - 1}{q - 1}$$

Geometric progression's summation
等比数列の和

Homework (Hand in next time pliz)

1. Expand $(x-1)^4$
2. What is the coefficient of x^9 in the expansion of $(1+x)^{13}$?
3. Simplify $\frac{\sqrt{45}}{\sqrt{5}}$ and $\frac{2}{1+\sqrt{3}}$
4. Find the roots of x^2+4x+2 and x^2+x+3
5. Compute the sum

$$S = \sum_{k=10}^{k=20} 2^k$$