MMA 数学特論 I。多項式系のアルゴリズム: グレブナー基底 & 消去法
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Practice test IV: Resultant

- You can use any theorem, proposition or corollary of the class lectures, just by citing its number inside the corresponding lecture: (example: "Lect II, Cor. 1" refers to the Corollary 1 of Lecture II, that is the Primitive Element Theorem).

Exercise 1 Write the Sylvester matrix of the polynomials A and B:

A	$2x^2+x-1$	$x-1$	$x^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0$
B	$30x^3+6x^2+x+1$	$x+1$	3

Exercise 2 Question 1 Are the following matrices in row echelon form ? *(the empty entries mean zero)*

$$A_1 = \begin{pmatrix} 4 & 1 & -1 & 1 & 0 \\ & & 4 & 1 & 1 \\ & & 1 & 0 & 0 \\ & & & & \\ & & & & \end{pmatrix} \qquad A_2 = \begin{pmatrix} 1 & 1 & -1 & 3 & 1 & 0 \\ & & 2 & 3 & -2 & -3 \\ & & & -1 & -2 & 3 \\ & & & & & 2 \\ & & & & & \\ & & & & & \end{pmatrix}$$

Exercise 3 Consider the 2 polynomials A and B,

$$A = 2x^2+x-1 \qquad B = 30x^3+6x^2+x+1.$$

We have $\mathsf{Res}(A,B) = 812$.

Given $n \in \mathbb{N}^\star$, we consider the map $\phi_n : \mathbb{Z}[X] \to \mathbb{Z}/n\mathbb{Z}[X]$, $\sum_i a_iX^i \mapsto \sum_i (a_i \bmod n)X^i$.

Question 1 Use Proposition 1 to compute:
$r_5 = \mathsf{Res}(\phi_5(A),\phi_5(B)) = \mathsf{Res}(A \bmod 5, B \bmod 5)$
$r_3 = \mathsf{Res}(\phi_3(A),\phi_3(B)) = \mathsf{Res}(A \bmod 3, B \bmod 3)$

Question 2 Can we use Proposition 2 to compute $r_2 = \mathsf{Res}(\phi_2(A),\phi_2(B)) = \mathsf{Res}(A \bmod 2, B \bmod 2)$? Compute anyway r_2.

Exercise 4 Let R be an integral domain (Lect. II, Def. 7... like $R = \mathbb{Z}$or $R = k[X,Y]$) Let $A, B, C \in R[X]$. Show the formula:

$$\mathsf{Res}(AB,C) = \mathsf{Res}(A,C)\mathsf{Res}(B,C).$$

(advice: use the formulas of Theorem 1, Slide 9)

Exercise 5 Consider the algebraic numbers $\alpha = (i+\sqrt{2})^{\frac{1}{7}} + (i+\sqrt{2})^{\frac{3}{7}} + (i+\sqrt{2})^{\frac{5}{7}} + 1$, and $\beta = (\sqrt{2}+3^{\frac{1}{3}})^2 + \sqrt{1+3^{\frac{1}{3}}}$.

The aim of this exercise is to compute a vanishing polynomial p_α of α and a vanishing polynomial p_β of β. Finally to compute $p_{\alpha\beta}$, a vanishing polynomial of the product $\alpha\beta$.

Question 1 Let $\alpha_1 = i + \sqrt{2}$. Compute a vanishing polynomial p_{α_1} of α_1. Deduce one for $\alpha_2 = \alpha^{\frac{1}{7}}$.

Question 2 Show that there exists a polynomial h such that $\alpha = h(\alpha_2)$. Use the resultant (Part II notes, n°2) to compute p_α.

Question 3 Compute a vanishing polynomial of $1 + 3^{\frac{1}{3}}$. Deduce one for $\sqrt{1+3^{\frac{1}{3}}}$.

Question 4 Use the resultant to compute a vanishing polynomial of $\beta_1 = \sqrt{2}+3^{\frac{1}{3}}$. Deduce one for $\beta_2 = \beta_1^2$.

Question 5 Use the resultant to compute p_β.

Question 6 Use the resultant to compute $p_{\alpha\beta}$.