MMA 数学特論 I。多項式系のアルゴリズム: グレブナー基底 & 消去法
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Practice test: Division equality and monomial ideals

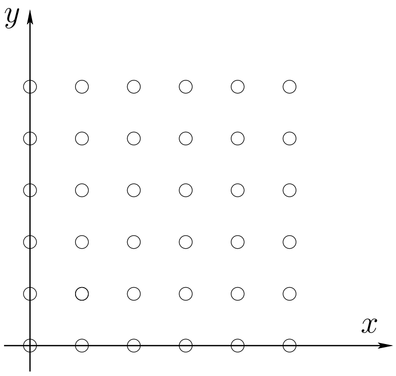
- You can use any theorem, proposition or corollary of the class lectures, just by citing its number inside the corresponding lecture: (example: "Lect II, Cor. 1" refers to the Corollary 1 of Lecture II, that is the Primitive Element Theorem).
- the answers are usually short. Write them directly on this sheet of paper.

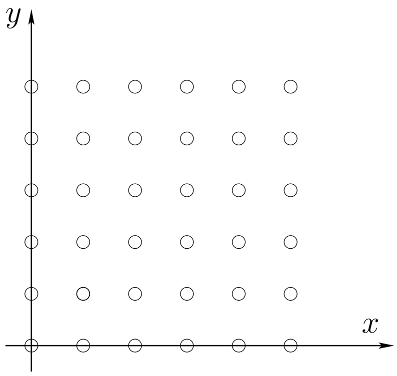
Exercise 1 Consider the division of $f = x^2y+xy^2+y^2$ by the sequence $[f_1 = xy-1\,,\ f_2 = y^2-1]$ for the order $y \prec_{lex} x$ (Exercise 5 of Handout 4):

$$f = (x+y)f_1 + 1.f_2 + x + y + 1,$$

hence, $a_1 = x+y$, $a_2 = 1$ and $r = x+y+1$ with the standard notations of the Lecture.

Question 1: Plot on the left-hand graphic below the associated Δ-sets Δ_1, Δ_2 and $\overline{\Delta}$ associated to $[f_1, f_2]$.





Question 2: For each monomial $m = x^{\alpha_1}y^{\alpha_2}$ occurring in a_1, plot a $\bullet$ on the point of coordinates $\mathsf{mdeg}(f_1) + (\alpha_1, \alpha_2) \in \mathbb{N}^2$ on the graphic. Then, plot a $\blacksquare$ on the point of coordinates $\mathsf{mdeg}(f_2) + (\beta_1, \beta_2) \in \mathbb{N}^2$ for each monomials $x^{\beta_1}y^{\beta_2}$ occurring in a_2, and finally put a $\blacklozenge$ for the monomials occurring in r.

Check the conclusion of Proposition 6.

Question 3: The division of f by the sequence $[f_2, f_1]$ gives a different result. We found $f = (x+1)f_2 + xf_1 + 2x + 1$.
Let us write $a'_1 := x+1$, $a'_2 = x$ and $r' = 2x+1$. On the right-hand graphic above, plot the Δ-sets Δ'_1, Δ'_2 and $\overline{\Delta}'$, corresponding to the sequence $[f_2, f_1]$.

For each monomials $x^{\alpha_1}y^{\alpha_2}$ occurring in a'_1 (respectively, in a'_2 and in r') plot the point of coordinates $\mathsf{mdeg}(f_2) + (\alpha_1, \alpha_2) \in \mathbb{N}^2$ (respectively of coordinates $\mathsf{mdeg}(f_1) + (\alpha_1, \alpha_2)$, and of corrdinates (α_1, α_2)) using a $\bullet$ (respectively a $\blacksquare$, and a $\blacklozenge$).

Again, check the conclusion of Proposition 6.

Question 4: On the right-hand graphic above, with a pen of different color, draw the Δ-sets Δ_1 and Δ_2 (the ones that you have drawn on the left-hand graphic).

Suppose now that the division of f by $[f_2, f_1]$ would be $[a_2, a_1]$ and r (*i.e.* $a'_1 = a_2, a'_2 = a_1, r' = r$ with the notations above, which is *not* true, just a supposition). Using Proposition 6, and the right-hand graphic, why this is not possible ?

Answer:

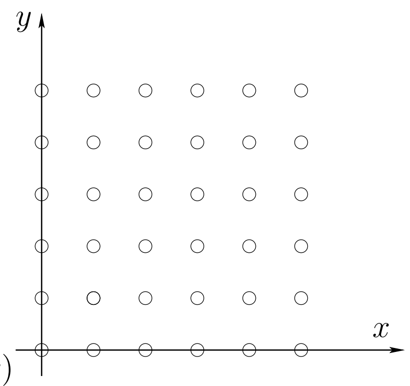
Exercise 2: In the Exercise 5 of the "Practical test" of May 20th, was considered the following polynomials:

$$f = x^3y^2 - x^2y^3 + 1 + x^2y^4 + x^4 - y + x^4y + x^5y + x^4y^3,$$

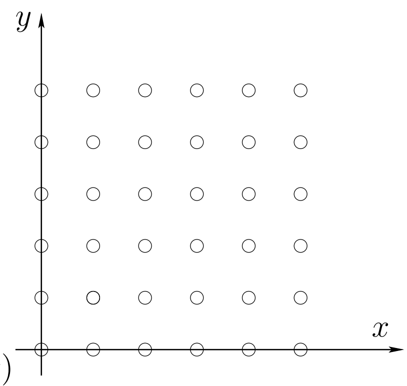
and $f_1 = x^3y^2 + x^4$, $f_2 = x^2y^3 - 1$.

Question 1: Plot the Δ-sets corresponding to the division of f by $[f_1, f_2]$.

For the *grevlex*(x, y) monomial order, the division of f by $[f_1, f_2]$ gave: $f = b_1f_1 + b_2f_2 + s$, with $b_1 = (xy + 1)$, $b_2 = (y - 1)$ and the remainder $s = x^4y$. Put the monomials on the left-hand side graphic below, in the same way it was asked in the previous exercise (use the marks $\bullet$, $\blacksquare$ and $\blacklozenge$).



grlex(x, y)



lex(x, y)

Question 2: Repeat Question 4 of Exercise 1 (*i.e.* (1) on the left-hand side graphic above, plot the Δ-sets, Δ'_1 ad Δ'_2 corresponding to the sequence $[f_2, f_1]$, with a pen of different color. (2) assume that $[b_2, b_1], s$ will be the division, so that the monomials are already plotted, by Question 1. (3) find a contradiction with Proposition 6. (4) check with Exercice 5 of Practice test 1)

Answer:

Question 3: Same question for the *lex*(x, y) monomial order. Why this shows that the division of f by $[f_1, f_2]$ or $[f_2, f_1]$ will give the *same* division (as was computed in Exercise 5 of Practice test 1)?

Answer:

Exercise 3 Let $f,\ f_1,\ f_2$ be three polynomials in $\Bbbk[X_1, \ldots, X_n]$, and $\prec$ a monomial order. Let $[a_1, a_2], r$ (respectively $[a'_1, a'_2], r'$) be the polynomials obtained by the division of f by $[f_1, f_2]$ (respectively by $[f_2, f_1]$).

When do we have $[a'_1, a'_2] = [a_2, a_1]$?

(advice 1: use Exercises 1 and 2.
advice 2: you need to find some *conditions* on the monomials occurring in a_1 and a_2. For example, something starting like *"Let X^α be a monomial occurring in a_1, and let $\alpha(1) := \mathsf{mdeg}_\prec(f_1) \in \mathbb{N}^n$. Then $\alpha + \alpha(1) \in ??$ and/or $\alpha + \alpha(1) \notin ??$ ..."*).

Answer:

Exercise 4 Corollary 2 of the Proposition 6, shows that the remainder r of a division by a sequence $[f_1, \ldots, f_s]$ depends *linearly* on the input polynomial f.

The proof shows that it is also true for each quotient a_i, $1 \leq i \leq s$.

Hence we have the following $s + 1$ endomorphisms of $\Bbbk$-vector spaces

$$\begin{array}{rcl} \varphi_i : \Bbbk[X_1, \ldots, X_n] & \to & \Bbbk[X_1, \ldots, X_n] \\ f & \mapsto & a_i \end{array} \quad \text{and} \quad \begin{array}{rcl} \varphi : \Bbbk[X_1, \ldots, X_n] & \to & \Bbbk[X_1, \ldots, X_n] \\ f & \mapsto & r \end{array}$$

Question: Are these maps one-one (injective, *i.e.* with kernels reduced to $\{0\}$) ?

What are the images of these linear maps ? (advice: just give a monomial basis of these vector spaces)

Answer:

Exercise 5 Prove the Corollary 2 of Lecture IV.
Answer:

Exercise 6 We consider an ideal $I = \langle f_1, \dots, f_s \rangle \subset \Bbbk[X_1, \dots, X_n]$, and a monomial order $\prec$.

Question 1 One property of the division, is: $a_i \neq 0 \Rightarrow \text{LM}_\prec(a_i f_i) \preceq \text{LM}_\prec(f)$ (Lect. III, Slide 18 Property (c)).

Let $\mathcal{I}(f) = \{i \mid \text{LM}_\prec(a_i f_i) = \text{LM}_\prec(f)\}$. Show that $\mathcal{I}(f) \neq \emptyset$.

Answer:

Question 2 Then show that $\text{LT}_\prec(f) = \sum_{i \in \mathcal{I}(f)} \text{LT}(a_i f_i)$.

Answer:

Question 3 Sometimes, there exists $f \in I$ such that the remainder r of the division by the sequence $[f_1, \dots, f_s]$ verifies $r \neq 0$.

Show that this is the case iff $\langle \text{LM}(f_1), \dots, \text{LM}(f_s) \rangle \subsetneq \langle \text{LM}(I) \rangle$.

Answer:

Exercise 7 Let $f \in \Bbbk[X_1, \dots, X_n]$ be a non-zero polynomial, and $\langle f \rangle$ the ideal that it generates in $\Bbbk[X_1, \dots, X_n]$.

Show that $\{f\}$ is a Gröbner basis of $\langle f \rangle$ for any monomial order $\prec$.

Answer:

Why $\{f\}$ is also a minimal Gröbner basis and a reduced Gröbner basis ?

Answer:

Exercise 8 Let G_1 and G_2 be two *minimal* Gröbner bases of an ideal I in $\Bbbk[X_1, \ldots, X_n]$ with respect to a monomial order $\prec$.

We denote by $\mathrm{LM}(G) = \mathrm{LM}_\prec(G) = \{\mathrm{LM}_\prec(g) \ : \ g \in G\}$. Show that $\mathrm{LM}(G_1) = \mathrm{LM}(G_2)$ (advice: Lemma 1 of Lect. IV will be useful)

Answer:

Exercise 9 Consider the following two polynomial systems F and H.

$$F \left| \begin{array}{l} f_1 = y^2 - 2y + 1, \\ f_2 = 2 + 3x + x^2 - 3y + xy - 5y^2 - 3xy^2. \end{array} \right.$$

$$H \left| \begin{array}{l} h_1 = -1 + 3y + x^2y + z - yz, \\ h_2 = 2x + 3y + 4z - 2yz + 3y^2z + x^2y^2z + yz^2 - 2y^2z^2, \\ h_3 = 3 + 6x + 2x^3 + 8z - 2xz + 4x^2z - 3z^2 - yz^2 + yz^3, \\ h_4 = -2x - 3y - 4z + yz + y^2z^2, \\ h_5 = 9 + 18x + 3x^2 + 12x^3 + 2x^5 + 21z - 12xz + 20x^2z - 4x^3z + 4x^4z - 18z^2 \\ \qquad\qquad +2xz^2 - 7x^2z^2 + 5z^3 - z^4. \end{array} \right.$$

Question 1: We <u>know</u> that F is a Gröbner basis of the ideal $\langle F \rangle$ it generates in $\mathbb{Q}[x, y]$ for the monomial order $lex(x, y)$ (*i.e.* $y \prec_{lex} x$, and $\langle \text{LM}(f_1), \text{LM}(f_2) \rangle = \text{LM}(\langle F \rangle)$).

Is F a *minimal* Gröbner basis ? If not, which polynomial can be removed ?

<u>Answer:</u>

Is the resulting minimal Gröbner basis *reduced* ?

<u>Answer:</u>

Question 2: Also, we <u>know</u> that H is a Gröbner basis of the ideal $\langle H \rangle$ it generates in $\mathbb{Q}[x, y, z]$ for the monomial order $grevlex(x, y, z)$ (*i.e.* $x \,_{grevlex}\!\succ y \,_{grevlex}\!\succ z$, and $\langle \text{LM}(h_1), \text{LM}(h_2), \text{LM}(h_3), \text{LM}(h_4), \text{LM}(h_5) \rangle = \text{LM}(\langle H \rangle)$).

Is H a *minimal* Gröbner basis ? If not which polynomial(s) can be removed ?

<u>Answer:</u>

Is the resulting minimal Gröbner basis *reduced* ?

<u>Answer:</u>