MMA 数学特論 I。多項式系のアルゴリズム: グレブナー基底 & 消去法

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Practice test: Monomial order and division algorithm

Exercise 1 For each polynomial f hereunder, write the leading monomial $\mathrm{LM}_\prec(f)$ and the multi-degree $\mathsf{mdeg}_\prec(f)$ with respect to the following monomial orders $\prec$.

(i) $f_1 = x^2 + y^5 + xz^4$

(ii) $f_2 = y^2 + yz + x^2$

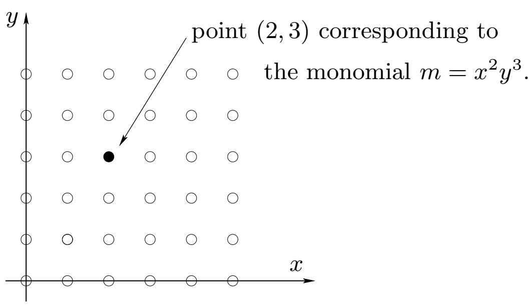
(iii) $f_3 = x^\alpha y^\beta z^\gamma - x^{\alpha-1} y^{\beta-2} z^{\gamma+3} + x^{\alpha-1} y^{\beta+2} z^{\gamma-1}$, for some $\alpha \geq 1$, $\beta \geq 2$ and $\gamma \geq 1$.

order $\prec$	$\mathsf{mdeg}_\prec(f_1)$	$\mathrm{LM}_\prec(f_1)$	$\mathsf{mdeg}_\prec(f_2)$	$\mathrm{LM}_\prec(f_2)$	$\mathsf{mdeg}_\prec(f_3)$	$\mathrm{LM}_\prec(f_3)$
$lex(x,y,z)$						
$lex(y,x,z)$						
$grlex(x,y,z)$						
$grlex(z,y,x)$						
$grevlex(x,y,z)$						
$grevlex(z,y,x)$						

Exercise 2: graphical representation of monomials

Given a monomial m of a polynomial algebra $\Bbbk[X_1,\ldots,X_n]$, its multi-degree $\mathsf{mdeg}(m) \in \mathbb{N}^n$ are the *coordinates* of a point in the n-space with axes $X_1, X_2, \ldots, X_n$.

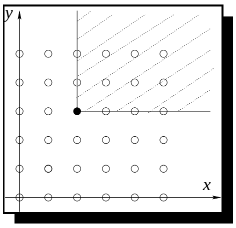
Example: (With $n = 2$, $X_1 = x$ and $X_2 = y$). Let $m = x^2y^3$ be the monomial of $\Bbbk[x,y]$. We have $\mathsf{mdeg}(m) = (2,3)$. The point corresponding to m in the plane (having for axes x, y) has coordinate $(2,3)$:



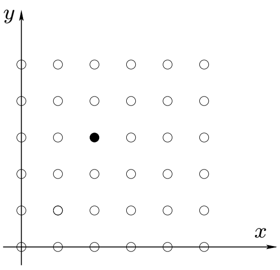
Each order on the monomials induce an order on the multi-integers and reciprocally:

$$\text{monomials: } x^{\alpha_1} y^{\alpha_2} \prec x^{\beta_1} y^{\beta_2} \iff \text{multi-integers: } (\alpha_1,\alpha_2) \prec (\beta_1,\beta_2).$$

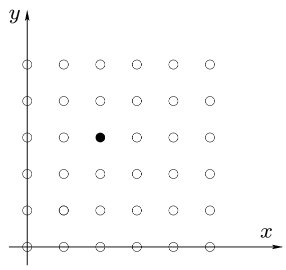
Question 1: Let $m = x^2y^3$ as above. For each of the 8 following orders $\prec$, draw the region defined by: $\boxed{\mathcal{M}(\prec m) := \{(\alpha,\beta) \in \mathbb{N}^2 \mid (\alpha,\beta) \prec (2,3)\}}$.



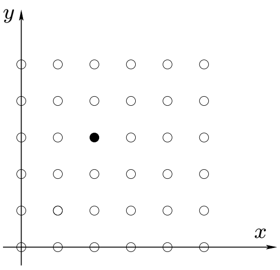
← Advice ! Please, draw like in this figure. In this example, the region shown is: $\{(\alpha,\beta) \in \mathbb{N}^2 \,|\, \alpha \geq 2 \text{ and } \beta \geq 3\}$.



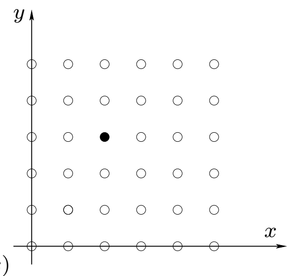
$\prec_{lex(x,y)}$



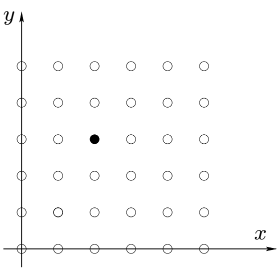
$\prec_{lex(y,x)}$



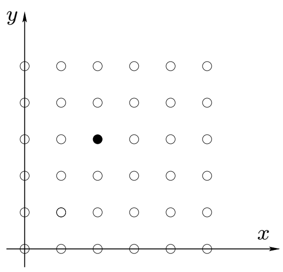
$\prec_{grlex(x,y)}$



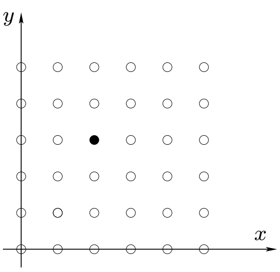
$\prec_{grlex(y,x)}$



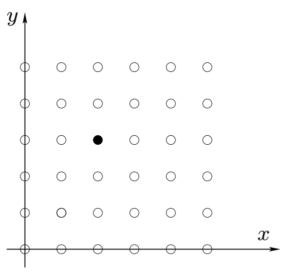
$\prec_{revlex(x,y)}$



$\prec_{revlex(y,x)}$



$\prec_{grevlex(x,y)}$



$\prec_{grevlex(y,x)}$

Question 2: For each of the 8 cases above, circle the largest monomial (for $\prec$) in $\mathcal{M}(\prec m)$.

Exercice 3 Show that the monomial orders *grevlex* and *grlex* are the same in two variables ($\iff$ prove the following equivalence):

$$\forall(\alpha,\beta,\gamma,\delta)\in\mathbb{N}^4,\qquad (x^\alpha y^\beta \prec_{grlex} x^\gamma y^\delta \iff x^\alpha y^\beta \prec_{grevlex} x^\gamma y^\delta).$$

Is it true in 3 variables (for example x, y, z) ?

$\rightarrow$ Answer:

Exercise 4 Let $\Bbbk[X_1,\ldots,X_n]$ be a polynomial in n variables over the field $\Bbbk$. Given to multi-integers $\alpha=(\alpha_1,\ldots,\alpha_n)$ and $\beta=(\beta_1,\ldots,\beta_n)$ we consider the two monomials X^α and X^β. We denote by $lex(X_1,\ldots,X_n)$ the lexicographic order for which $X_1 \succ X_2 \succ \cdots \succ X_n$ (while for example $lex(X_n,X_{n-1},\ldots,X_1)$ corresponds to $X_n \succ X_{n-1} \succ \cdots \succ X_1$).

Show that: $$X^\alpha \prec_{revlex(X_1,\ldots,X_n)} X^\beta \iff X^\alpha \;{}_{lex(X_n,\ldots,X_1)}\!\succ X^\beta.$$

$\rightarrow$ Answer:

Exercise 5 Consider the following polynomial in $\Bbbk[x, y, z]$. In each of the 4 following cases, compute the division of f by the sequence $[f_1, f_2]$, as indicated.

!!: You need to rewrite f with its monomials *ordered* according to first: $grlex(x, y)$ (cases 1 and 2) and to, second: $lex(x, y)$ (cases 3 and 4).

$$f = x^3y^2 - x^2y^3 + 1 + x^2y^4 + x^4 - y + x^4y + x^5y + x^4y^3$$

$\boxed{grlex(x,y)}$ $a_1 :$ $r :$

$a_2 :$

$f_1 = x^3y^2 + x^4$

$f_2 = x^2y^3 - 1$

$\boxed{grlex(x,y)}$ $a_1 :$ $r :$

$a_2 :$

$f_1 = x^2y^3 - 1$

$f_2 = x^3y^2 + x^4$

The remainder and the quotients a_1 and a_2 should be different in the above 2 cases (we have just changes the $[f_1, f_2]$ to $[f_2, f_1]$).

For the lex order, however, you should find the <u>same</u> results whatever we take $[f_1, f_2]$ or $[f_2, f_1]$.

$$f = x^3y^2 - x^2y^3 + 1 + x^2y^4 + x^4 - y + x^4y + x^5y + x^4y^3$$

!!: You need to rewrite the monomials of f in decreasing order for $lex(x, y)$.

$\boxed{lex(x,y)}$ a_1 : r :

a_2 :

$f_1 = x^2y^3 - 1$

$f_2 = x^4 + x^3y^2$

$\boxed{lex(x,y)}$ a_1 : r :

a_2 :

$f_1 = x^4 + x^3y^2$

$f_2 = x^2y^3 - 1$