MMA 数学特論 I。多項式系のアルゴリズム: グレブナー基底 & 消去法
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Practice test V: Elimination and Nullstellensatz

- **!!!** Another part of the test is to do directly in Mathematica **!!!**

Exercise 1 (radical ideal) We defined the radical of an ideal for a polynomial algebra (Lect. VII, Def. 4). But the definition is the same over any ring R:

$$\text{for } I \text{ ideal of } R, \qquad \sqrt{I} = \{x \in R \mid \exists n \in \mathbb{N},\ x^n \in I\}.$$

Question 1 Show that $\sqrt{I}$ is an ideal of R (for a definition Lect. II, Def. 3)
Answer:

Question 2 Given two ideals I and J of R, we know that $I \cap J$ is an ideal of R. Show that $\sqrt{I \cap J} = \sqrt{I} \cap \sqrt{J}$.
Answer:

Exercise 2 (Don't use a computer: computations are easy) Consider the following polynomial system:

$$\begin{aligned} x^2 + 2y^2 &= 3 \\ x^2 + xy + y^2 &= 3 \end{aligned}$$

Question 1 Find all the solution in $\mathbb{C}^2$ (write the computations you did briefly)

Answer:

Question 2 Write down the solutions in $\mathbb{Q}^2$. What is the smallest field containing all the solutions ?

Answer:

Exercise 3 The aim of this exercise is to show that given a point $a = (a_1, \dots, a_n) \in k^n$ for a field k, the vanishing ideal $\mathbf{I}_k(\{a\}) \subset k[X_1, \dots, X_n]$ is equal to $\langle X_1 - a_1, \dots, X_n - a_n \rangle$

Question 1 Show that the polynomials $P_i(X_1, \dots, X_n) = X_i - a_i$ are all in $\mathbf{I}_k(\{a\})$ for $i = 1, \dots, n$.

Answer:

Question 2 Let $f \in \mathbf{I}_k(\{a\})$. Choose a monomial order $\prec$ on the monomials in $X_1, \dots, X_n$. What is $\textsc{lt}_\prec(P_i)$?

Answer:

Question 3 Let r be the remainder of the division of f by the sequence $[P_1, \dots, P_n]$. Prove that r is a constant.

Answer:

Question 4 Show that $r = 0$ is the only possibility. Deduce that $\{P_1, \dots, P_n\}$ is a (Gröbner) basis of $\mathbf{I}_k(\{a\})$.

Answer: