MMA Advanced Lecture I **Handout 4**

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This handout is available from my webpage:

http://yokoemon.web.fc2.com/education.html

Now planning to move.

Announcement

- $\boxed{\textbf{HOMEWORK SUBMISSION}}$ or given by TEST. Please note.
- Slides of Lecture III are (re-)updated, so please download if you want. Today's slides (given in the previous class) are not yet, but will be soon.

Warming Up to do homework, check understanding.

1. Check the following inequalities, where the order on the variable is implicitely $X_1 \succ X_2 \succ X_3$:
   (a) $X_1^3X_2^2X_3 \prec_{lex} X_1^3X_2^2X_3^4$
   (b) $X_1X_2X_3^5 \prec_{grlex} X_1X_2^2X_3^4$
   (c) $X_1^4X_2X_3^3 \prec_{grevlex} X_1X_2^5X_3^2$
2. Rewriting of the following polynomial
$$f(X,Y,Z) = 4XY^2Z + 4Z^2 - 5X^3 + 7X^2Z^2$$
   ordering the terms using the...
   (a) lex order: $f = -5X^3 + 7X^2Z^2 + 4XY^2Z + 4Z^2$
   (b) grlex order: $f = 7X^2Z^2 + 4XY^2Z - 5X^3 + 4Z^2$
   (c) grevlex order: $f = 4XY^2Z + 7X^2Z^2 - 5X^3 + 4Z^2$
   (Can you check the above using Mathematica ?)
3. Let f as before (#2) and let $\succ$ be $\succ_{lex}$,
   (a) Multi-degree $\mathrm{mdeg}(f) = (3,0,0)$
   (b) Leading coefficient $\mathrm{LC}(f) = -5$
   (c) Leading monomial $\mathrm{LM}(f) = X^3$
   (d) Leading term $\mathrm{LT}(f) = -5X^3$

4. Dividing $f = xy^2 + 1$ by $f_1 = xy + 1$ and $f_2 = y + 1$ (using lex order with $x \succ y$), check the following division:
$$xy^2 + 1 = y(xy + 1) + (-1)(y + 1) + 2$$

$a_1 :$

$a_2 :$

$f_1 = xy + 1$ | $xy^2 + 1$

$f_2 = y + 1$

5. Do and check the following (explained on the 5/13's blackboard):

   (a) Divide $f = x^2y + xy^2 + y^2$ by $f_1 = xy - 1$ and $f_2 = y^2 - 1$, using the lex order with $x \succ y$. # **Solution is :** $f = (x + y)(xy - 1) + 1 \cdot (y^2 - 1) + x + y + 1$

   (b) Divide f by $f_1 = y^2 - 1$ and $f_2 = xy - 1$, using the lex order with $x \succ y$. This is the same as (a) except that we have changed the order of the divisors.

   # **Solution is :** $f = (x + 1)(y^2 - 1) + x(xy - 1) + 2x + 1$

   (c) Check that the remainder of (b) is <u>**different**</u> from what we got in (a). [1]

$a_1 :$

$a_2 :$

$f_1 = xy - 1$ | $x^2 + xy^2 + y^2$

$f_2 = y^2 - 1$

$a_1 :$

$a_2 :$

$f_1 = y^2 - 1$ | $x^2y + xy^2 + y^2$

$f_2 = xy - 1$

[1] Thus, we must conclude that the division algorithm is an imperfect generalization of its univariate counterpart. To remedy this situation, in dealing with a collection of polynomials $f_1, ..., f_s \in k[x_1, ..., x_n]$, it is frequently desirable to pass to the ideal I they generate. This allows the possibility of going from $f_1, ..., f_s$ to a different generating set for I. So we can still ask whether there might be a good generating set for I. For such a set, we would want the remainder r on division by the good generators to be uniquely determined and the condition $r = 0$ should be equivalent to membership in the ideal. **Gröbner basis** have exactly these good properties.